

# TUNNELING OF THE KAWASAKI DYNAMICS AT LOW TEMPERATURES IN TWO DIMENSIONS

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**ABSTRACT.** Consider a lattice gas evolving according to the conservative Kawasaki dynamics at inverse temperature  $\beta$  on a two dimensional torus  $\Lambda_L = \{0, \dots, L-1\}^2$ . We prove the tunneling behavior of the process among the states of minimal energy. More precisely, assume that there are  $n^2 \ll L$  particles and that the initial state is the configuration in which all sites of the square  $\mathbf{x} + \{0, \dots, n-1\}^2$  are occupied. We show that in the time scale  $e^{2\beta}$  the process is close to a Markov process on  $\Lambda_L$  which jumps from any site  $\mathbf{x}$  to any other site  $\mathbf{y} \neq \mathbf{x}$  at a strictly positive rate which can be expressed in terms of the jump rates of simple random walks.

## 1. INTRODUCTION

We introduced recently [2] a general method to prove the metastable behavior [12, 24] of reversible Markov processes on countable state spaces. The procedure relies on potential theory, as the approach proposed previously by Bovier et al. in [8, 9]. We applied the ideas introduced in [2] to derive in [3] the asymptotic behavior of the condensated [1, 13, 16, 20] in supercritical reversible zero range processes, and to describe in [19] the evolution in the ergodic time scale of a random walk among random traps [5, 6, 7]. More recently, to illustrate the power of the method in a simple context, we derived in [4] the metastable behavior of Markov processes on finite state spaces whose jump rates satisfy two natural, and almost necessary, assumptions for metastability. This class includes to our knowledge all examples of Markov processes in *fixed and finite* state spaces for which metastability has been proved. It includes, in particular, the much studied Ising lattice gas evolving according to Glauber dynamics on a finite space [21, 22] or evolving locally according to the Kawasaki dynamics [18, 14, 17], in both cases in any dimension. The result presented in [4] asserts the existence of tunneling behaviors at different time scales, and provides a recursive algorithm to determine the metastable sets, the time scales in which the tunneling is observed, and the asymptotic Markovian dynamics which describes the evolution of the process among the metastable sets. An explicit expression for the metastable sets, the time scales and the asymptotic regimes have to be worked out in each case, and may be a difficult task.

While for the Glauber dynamics the challenge is to depict the passage at very low temperatures from a metastable state to the stable state through a nucleation process, for the conservative Kawasaki dynamics one is not only interested in describing the nucleation phase at very low temperatures, but also in analyzing the time evolution of the process after the nucleation has occurred, when the process

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is expected to evolve, in an appropriate time scale, by jumping from one state of minimal energy to another.

To clarify this last sentence and to present the main result of this article, denote by  $\Lambda_L$  the two dimensional torus of length  $L$  and fix an integer  $n \ll L$ . Consider  $n^2$  particles evolving on  $\Lambda_L$  according to the Kawasaki dynamics at inverse temperature  $\beta$ . The states of minimal energy are the  $L^2$  configurations  $\eta^{\mathbf{x}}$ ,  $\mathbf{x} \in \Lambda_L$ , obtained when the particles form squares of length  $n$ . Here,  $\eta^{\mathbf{x}}$  is the configuration in which a site is occupied if and only if it belongs to the square  $\mathbf{x} + \{0, \dots, n-1\}^2$ . Assume that the initial configuration is  $\eta^{\mathbf{x}}$ . The main result of this article states that on the time scale  $e^{2\beta}$ , for  $\beta$  large the process is close to a Markov process on  $\Lambda_L$  which jumps from  $\mathbf{x}$  to  $\mathbf{y}$  at a positive rate denoted by  $r(\mathbf{x}, \mathbf{y})$ . The rates  $r(\mathbf{x}, \mathbf{y})$  depend obviously on the parameters  $n$  and  $L$  and can be expressed in terms of the jump probabilities of two-dimensional simple random walks. It is important to stress that the asymptotic dynamics is *non-local*:  $r(\mathbf{x}, \mathbf{y}) > 0$  for all  $\mathbf{x} \neq \mathbf{y} \in \Lambda_L$ .

Denote by  $r_{L,n}(\mathbf{x}, \mathbf{y})$  the asymptotic rates to emphasize their dependence on the parameters  $L$  and  $n$ . It is not difficult to compute the limiting behavior of the rates  $r_{L,n}$  as  $L \uparrow \infty$ . It is less clear how to express the dependence of these rates on the number  $n^2$  of particles. This is left to a future work, as well as the extension to three dimensions or the interesting case in which the size  $L$  of the torus or the number  $n^2$  of particles increase with  $\beta$ . Another interesting question concerns the behavior of the process before the final nucleation, when two squares have been formed and evolve as two coalescing random walks on  $\Lambda_L$ .

To prove the main result of this article, informally presented above, one has to describe precisely the typical excursions performed by the process when going from a minimal energy configuration  $\eta^{\mathbf{x}}$  to another minimal energy configuration  $\eta^{\mathbf{y}}$ . In our case, the typical path consists in shifting sequentially a row or a column of particles along the sides of the square, a procedure which appeared in a different context in [14].

Metastability of locally conserved dynamics or of conservative dynamics superposed with non-conservative ones have been considered before. Peixoto [25] examined the metastability of the two dimensional Ising lattice gas at low temperature evolving according to a superposition of the Glauber dynamics with a stirring dynamics. Den Hollander et al. [18] and Gaudillière et al. [14] described the critical droplet, the nucleation time and the typical trajectory followed by the process during the transition from a metastable state to the stable state in a two dimensional Ising lattice gas evolving under the Kawasaki dynamics at very low temperature in a finite square in which particles are created and destroyed at the boundary. This result has been extended to three dimensions by den Hollander et al. [17]. Using the potential theoretical approach introduced in [8, 9], Bovier et al. [10] presented the detailed geometry of the set of critical droplets and provided sharp estimates for the expectation of the nucleation time for this model in dimension two and three.

More recently, Gaudillière et al. [15] proved that the dynamics of particles evolving according to the Kawasaki dynamics at very low temperature and very low density in a two-dimensional square whose length increases as the temperature decreases can be approximated by the evolution of independent particles. These results were used in [11], together with the potential theoretical approach alluded to above, to obtain sharp estimates for the expectation of the nucleation time for this model.

## 2. NOTATION AND RESULTS

We consider a lattice gas on a torus subjected to a Kawasaki dynamics at inverse temperature  $\beta$ . Let  $\Lambda_L = \{1, \dots, L\}^2$ ,  $L \geq 1$ , be a square with periodic boundary conditions. Denote by  $\Lambda_L^*$  the set of bonds of  $\Lambda_L$ . This is the set of unordered pairs  $\{x, y\}$  of  $\Lambda_L$  such that  $|x - y| = 1$ , where  $|\cdot|$  stands for the Euclidean distance. The configurations are denoted by  $\eta = \{\eta(x) : x \in \Lambda_L\}$ , where  $\eta(x) = 1$  if site  $x$  is occupied and  $\eta(x) = 0$  if site  $x$  is vacant. The Hamiltonian  $\mathbb{H}$ , defined on the state space  $\Omega_L = \{0, 1\}^{\Lambda_L}$ , is given by

$$-\mathbb{H}(\eta) = \sum_{\{x, y\} \in \Lambda_L^*} \eta(x)\eta(y) .$$

The Gibbs measure at inverse temperature  $\beta$  associated to the Hamiltonian  $\mathbb{H}$ , denoted by  $\mu_\beta$ , is given by

$$\mu_\beta(\eta) = \frac{1}{Z_\beta} e^{-\beta \mathbb{H}(\eta)} ,$$

where  $Z_\beta$  is the normalizing partition function.

We consider the continuous-time Markov chain  $\{\eta_t^\beta : t \geq 0\}$  on  $\Omega_L$  whose generator  $L_\beta$  acts on functions  $f : \Omega_L \rightarrow \mathbb{R}$  as

$$(L_\beta f)(\eta) = \sum_{\{x, y\} \in \Lambda_L^*} c_{x, y}(\eta) [f(\sigma^{x, y} \eta) - f(\eta)] ,$$

where  $\sigma^{x, y} \eta$  is the configuration obtained from  $\eta$  by exchanging the occupation variables  $\eta(x)$  and  $\eta(y)$ :

$$(\sigma^{x, y} \eta)(z) = \begin{cases} \eta(z) & \text{if } z \neq x, y, \\ \eta(y) & \text{if } z = x, \\ \eta(x) & \text{if } z = y. \end{cases}$$

The rates  $c_{x, y}$  are given by

$$c_{x, y}(\eta) = \exp \left\{ -\beta [\mathbb{H}(\sigma^{x, y} \eta) - \mathbb{H}(\eta)]_+ \right\} ,$$

and  $[a]_+$ ,  $a \in \mathbb{R}$ , stands for the positive part of  $a$ :  $[a]_+ = \max\{a, 0\}$ . We sometimes represent  $\eta_t^\beta$  by  $\eta^\beta(t)$  and we frequently omit the index  $\beta$  of  $\eta_t^\beta$ .

A simple computation shows that the Markov process  $\{\eta_t : t \geq 0\}$  is reversible with respect to the Gibbs measures  $\mu_\beta$ ,  $\beta > 0$ , and ergodic on each irreducible component formed by the configurations with a fixed total number of particles. Let  $\Omega_{L, K} = \{\eta \in \Omega_L : \sum_{x \in \Lambda_L} \eta(x) = K\}$ ,  $0 \leq K \leq |\Lambda_L|$ , and denote by  $\mu_{\beta, K}$  the Gibbs measure  $\mu_\beta$  conditioned on  $\Omega_{L, K}$ :

$$\mu_{\beta, K}(\eta) = \frac{1}{Z_{\beta, K}} e^{-\beta \mathbb{H}(\eta)} , \quad \eta \in \Omega_{L, K} ,$$

where  $Z_{\beta, K}$  is the normalizing constant  $Z_{\beta, K} = \sum_{\eta \in \Omega_{L, K}} \exp\{-\beta \mathbb{H}(\eta)\}$ .

For each configuration  $\eta \in \Omega_{L, K}$ , denote by  $\mathbf{P}_\eta^\beta$  the probability on the path space  $D([0, \infty), \Omega_{L, K})$  induced by the Markov process  $\{\eta_t : t \geq 0\}$  starting from  $\eta$ . Expectation with respect to  $\mathbf{P}_\eta^\beta$  is represented by  $\mathbf{E}_\eta^\beta$ .

Assume that  $K = n^2$  for some  $4 \leq n \ll L$ , and denote by  $Q$  the square  $\{0, \dots, n-1\} \times \{0, \dots, n-1\}$ . For  $\mathbf{x} \in \Lambda_L$ , let  $Q_{\mathbf{x}} = \mathbf{x} + Q$  and let  $\eta^{\mathbf{x}}$  be the configuration

in which all sites of the square  $Q_{\mathbf{x}}$  are occupied. Denote by  $\Omega^0 = \Omega_{L,K}^0$  the set of square configurations:

$$\Omega^0 = \{\eta^{\mathbf{x}} : \mathbf{x} \in \Lambda_L\}.$$

We claim that the ground states of the energy  $\mathbb{H}$  in  $\Omega_{L,K}$  are the square configurations:

$$\min_{\eta \in \Omega_{L,K}} \mathbb{H}(\eta) = \mathbb{H}(\eta^{\mathbf{x}}) = -2n(n-1),$$

and  $\mathbb{H}(\eta) > -2n(n-1)$  for all  $\eta \in \Omega_{L,K} \setminus \Omega^0$ .

To prove this claim, fix a configuration  $\eta \in \Omega_{L,K}$ . Since  $n^2 < L$ , there exists a configuration  $\xi$  in  $\{0,1\}^{\mathbb{Z}^2}$  whose energy coincides with the energy of  $\eta$ . Associate to each particle of  $\xi$  a  $1 \times 1$  square centered at the site occupied by the particle. Consider the smallest rectangle in  $\mathbb{Z}^2$  which contains all squares. We may assume that each row and column of the rectangle contains at least one square. If this is not the case, by translating simultaneously some squares, we obtain from  $\xi$  a new configuration  $\xi'$  whose energy is strictly smaller than the one of  $\xi$  and with the required property.

Denote by  $m_1 \leq m_2$  the lengths of the smallest rectangle which contains all squares. The area of the rectangle, equal to  $m_1 m_2$ , must be larger than or equal to the number of particles  $n^2$ . It follows from this inequality that  $m_1 + m_2 \geq 2n$ , with an equality if and only if  $m_1 = m_2 = n$ . Since each row and each column contains at least a square, there exist at least  $2(m_1 + m_2)$  bonds between a particle and a hole.

Since there are  $n^2$  particles, if all 4 bonds of each particle were attached to another particle, the energy would be  $-2n^2$ . For the configurations  $\eta'$ , we have seen that  $2(m_1 + m_2)$  bonds link a particle to a hole. Hence, the energy of this configuration is at least  $-(2n^2 - m_1 - m_2) \geq -2n(n-1)$ , with an equality if and only if  $m_1 = m_2 = n$ , i.e., if  $\xi'$  is a square configuration. This proves the claim.

We examine in this article the asymptotic evolution of the Markov process  $\{\eta_t : t \geq 0\}$  among the  $|\Lambda_L|$  ground states  $\{\eta^{\mathbf{x}} : \mathbf{x} \in \Lambda_L\}$  as the temperature vanishes. The main theorem of this article reads as follows. Recall the definition of Tunneling introduced in [2].

**Theorem 2.1.** *The sequence of Markov processes  $\{\eta_t^\beta : t \geq 0\}$  exhibits a tunneling behavior on the time-scale  $e^{2\beta}$ , with metastates  $\{\{\eta^{\mathbf{x}}\} : \mathbf{x} \in \Lambda_L\}$ , metapoints  $\eta^{\mathbf{x}}$  and asymptotic Markov dynamics characterized by strictly positive rates  $r(\eta^{\mathbf{x}}, \eta^{\mathbf{y}})$ .*

Denote by  $\{\eta_t^{\mathcal{F}} : t \geq 0\}$  the trace of the process  $\eta_t$  on the set of ground states  $\Omega^0$ . We refer to [2] for a precise definition of the trace process. Theorem 2.1 asserts that starting from a ground state the speeded up process  $\eta^{\mathcal{F}}(e^{2\beta}t)$  converges to a Markov process on  $\Omega^0$  which jumps from  $\eta^{\mathbf{x}}$  to  $\eta^{\mathbf{y}}$  at rate  $r(\eta^{\mathbf{x}}, \eta^{\mathbf{y}})$ . Moreover, in this time scale the time spent outside  $\Omega^0$  is negligible: for every  $\mathbf{x} \in \Lambda_L$ ,  $t > 0$ ,

$$\lim_{\beta \rightarrow \infty} \mathbf{E}_{\eta^{\mathbf{x}}}^\beta \left[ \int_0^t \mathbf{1}_{\{\eta(e^{2\beta}s) \notin \Omega^0\}} ds \right] = 0. \quad (2.1)$$

**Remark 2.2.** *The asymptotic rates  $r(\eta^{\mathbf{x}}, \eta^{\mathbf{y}})$  depend, naturally, on the parameters  $L$  and  $n$ . We stress that these rates are strictly positive. The asymptotic behavior is therefore non local, the limit process being able to jump from a configuration  $\eta^{\mathbf{x}}$  to any configuration  $\eta^{\mathbf{y}}$  with a positive probability. We expect the rate to decrease with  $n$  and with the distance between  $\mathbf{x}$  and  $\mathbf{y}$ . We present in Corollary 6.3 an explicit*

formula for these rates in terms of the transition probability of simple random walks. We shall investigate in a future work the dependence of the rates on  $n$  and on the distance from  $\mathbf{x}$  to  $\mathbf{y}$  as  $L \uparrow \infty$ .

**Remark 2.3.** *The Kawasaki dynamics is a perfect example to illustrate the difference between what has been defined in [2] as tunneling and metastability.*

*In the definition of tunneling nothing is required if the process starts from a configuration which does not belong to the set  $\Omega^0$  of ground states. In contrast, for a process to be metastable in the context delimited in Theorem 2.1, one has to show that the time spent outside the set of ground states is negligible in the sense (2.1) for any starting point.*

*This is not the case for the Kawasaki dynamics. Suppose, for instance, that  $n^2 = n_1^2 + n_2^2$ ,  $n_2 > n_1 \geq 4$ . This condition is not important and the same observation can be made writing  $n^2$  as the sum of the area of four rectangles. Consider the configuration where the occupied sites form two squares of side  $n_1$  and  $n_2$ , respectively, at distance at least 2. As for a ground state, the process remains in this configuration for an exponential time of order  $e^{2\beta}$  which implies that the process is not metastable in the sense of [2, Definition 3.7].*

*To prove metastability we would need to incorporate in the analysis a large class of configurations, never visited if the process starts from a ground state.*

In distinction with the Glauber dynamics, where the valleys have depths of several different orders, in the Kawasaki dynamics the depth of a valley is either of order  $e^\beta$  or of order  $e^{2\beta}$ . These valleys may, of course, lie at different levels, where by level of a valley we mean the inverse of the order of magnitude of the measure of the valley. More precisely, if  $\mathcal{E} \subset \Omega_{L,K}$  is the basin of a valley, the level of the valley is  $\lim_{\beta \rightarrow \infty} \beta^{-1} \log\{\mu_\beta(\eta^{\mathbf{x}})/\mu_\beta(\mathcal{E})\}$ .

### 3. SKETCH OF THE PROOF

The proof of Theorem 2.1 relies on the strategy presented in [4] to prove the metastability of reversible Markov processes evolving on finite state spaces. A simple computation shows that assumptions (2.1) and (2.2) of that article are satisfied. Indeed, since  $\mathbb{H}(\sigma^{x,y}\eta) - \mathbb{H}(\eta) = (\eta_y - \eta_x)\{\sum_{|z-y|=1} \eta_z - \sum_{|z-x|=1} \eta_z + \eta_y - \eta_x\}$ , the jump rates  $c_{x,y}(\eta)$  may only assume the values 1,  $e^{-\beta}$ ,  $e^{-2\beta}$  and  $e^{-3\beta}$ , which proves assumptions (2.1) and (2.2).

Denote by  $R_\beta(\eta, \xi)$  the rate at which the process  $\eta_t$  jumps from  $\eta$  to  $\xi$  so that  $R_\beta(\eta, \xi) = c_{x,y}(\eta)$  if  $\xi = \sigma^{x,y}\eta$  for some bond  $\{x, y\} \in \Lambda_L^*$ , and  $R_\beta(\eta, \xi) = 0$ , otherwise.

A self-avoiding path  $\gamma$  from  $A$  to  $B$ ,  $A, B \subset \Omega_{L,K}$ ,  $A \cap B = \emptyset$ , is a sequence of configurations  $(\xi_0, \dots, \xi_n)$  such that  $\xi_0 \in A$ ,  $\xi_n \in B$ ,  $\xi_i \neq \xi_j$ ,  $i \neq j$ ,  $R_\beta(\xi_j, \xi_{j+1}) > 0$ ,  $0 \leq j < n$ . Denote by  $\Gamma_{A,B}$  the set of self-avoiding paths from  $A$  to  $B$  and let

$$G_\beta(A, B) := \max_{\gamma \in \Gamma_{A,B}} G_\beta(\gamma), \quad G_\beta(\gamma) := \min_{0 \leq i < n} \mu_\beta(\xi_i) R_\beta(\xi_i, \xi_{i+1})$$

if  $\gamma = (\xi_0, \dots, \xi_n)$ . Since  $\mu_\beta(\xi_i) R_\beta(\xi_i, \xi_{i+1}) = \min\{\mu_\beta(\xi_i), \mu_\beta(\xi_{i+1})\}$ ,  $G_\beta(\gamma) = \min_{0 \leq i \leq n} \mu_\beta(\xi_i)$  and  $G_\beta(A, B)$  is the measure of the saddle configuration from  $A$  to  $B$ .

Denote by  $D_\beta$  the Dirichlet form associated to the generator of the Markov process  $\eta_t$ :

$$D_\beta(f) = \frac{1}{2} \sum_{\{x,y\} \in \Lambda_L^*} \sum_{\xi \in \Omega_{L,K}} \mu_\beta(\xi) c_{x,y}(\xi) \{f(\sigma^{x,y}\xi) - f(\xi)\}^2, \quad f : \Omega_{L,K} \rightarrow \mathbb{R}.$$

Let  $\text{cap}_\beta(A, B)$ ,  $A, B \subset \Omega_{L,K}$ ,  $A \cap B = \emptyset$ , be the capacity between  $A$  and  $B$ :

$$\text{cap}_\beta(A, B) = \inf_f D_\beta(f),$$

where the infimum is carried over all functions  $f : \Omega_{L,K} \rightarrow \mathbb{R}$  such that  $f(\xi) = 1$  for all  $\xi \in A$ , and  $f(\xi) = 0$  for all  $\xi \in B$ . We proved in [4, Lemma 4.2 and 4.3] that the ratio  $\text{cap}_\beta(A, B)/G_\beta(A, B)$  converge as  $\beta \uparrow \infty$ : For every  $A, B \subset \Omega_{L,K}$ ,  $A \cap B = \emptyset$ ,

$$\lim_{\beta \rightarrow \infty} \frac{\text{cap}_\beta(A, B)}{G_\beta(A, B)} = C(A, B) \in (0, \infty). \quad (3.1)$$

We claim that  $G_\beta(\{\eta^{\mathbf{x}}\}, \{\eta^{\mathbf{y}}\}) = e^{-2\beta} \mu_\beta(\eta^{\mathbf{x}})$  for  $\mathbf{x} \neq \mathbf{y}$ . Denote by  $e_1, e_2$  the canonical basis of  $\mathbb{R}^2$ . On the one hand, any path  $\gamma$  from  $\eta^{\mathbf{x}}$  to a set  $A \not\ni \eta^{\mathbf{x}}$  is such that  $G_\beta(\gamma) \leq e^{-2\beta} \mu_\beta(\eta^{\mathbf{x}})$ . On the other hand, it is easy to construct a self-avoiding path  $\gamma = (\eta^{\mathbf{x}} = \xi_0, \dots, \xi_n = \eta^{\mathbf{x}+e_i})$  from  $\eta^{\mathbf{x}}$  to  $\eta^{\mathbf{x}+e_i}$ , and therefore a path from  $\eta^{\mathbf{x}}$  to  $\eta^{\mathbf{y}}$ , such that  $\mu_\beta(\xi_j) \geq e^{-2\beta} \mu_\beta(\eta^{\mathbf{x}})$ ,  $0 \leq j \leq n$ . This proves the claim.

It follows from the previous identity and from (3.1) that  $\text{cap}_\beta(\{\eta^{\mathbf{x}}\}, \{\eta^{\mathbf{y}}\})$  is of order  $e^{-2\beta} \mu_\beta(\eta^{\mathbf{x}})$ . In particular, to examine the evolution of the process  $\eta_t$  among the competing metastable states  $\eta^{\mathbf{x}}$  we need only to care of the states whose measure are greater than or equal to  $e^{-2\beta} \mu_\beta(\eta^{\mathbf{x}})$ . Actually, only a much smaller class is relevant for the problem.

Consider the configuration  $\eta^{\mathbf{x}}$ . There are 8 jumps of rate  $e^{-2\beta}$  involving the particles at the corner of the square, all the other jumps being of rate  $e^{-3\beta}$ . These latter ones can be neglected since we are interested in the asymptotic dynamics as the temperature vanishes. After a jump of the particle at the corner, this detached particle performs a rate one, symmetric, nearest-neighbor random walk on the torus  $\Lambda_L$  until it attains the boundary of the set  $Q_{\mathbf{x}} \setminus \{\mathbf{z}\}$ , where  $\mathbf{z}$  is the corner which lost a particle.

The detached particle may have reached the boundary at  $\mathbf{z}$ , in which case the process returned to its original position. It may also have touched the set  $Q_{\mathbf{x}} \setminus \{\mathbf{z}\}$  at one of its sides. There are sixteen different types of possible configurations, reflecting the four possible corners and the four possible sides.

Each possible type corresponds to a valley of depth  $e^\beta$ , denoted in the next section by  $\mathcal{E}_{\mathbf{x}}^{i,j}$ ,  $0 \leq i, j \leq 3$ . At this point we analyze the behavior of the process  $\eta_t$  starting from one of these sets to find out that after an asymptotic exponential time of mean  $e^\beta$ , the process may reach the configuration  $\eta^{\mathbf{x}}$ , a valley  $\mathcal{E}_{\mathbf{x}}^{i,j}$  or a new class of valleys represented by  $\{\eta_{\mathbf{x}}^{a,(\mathbf{k},\ell)}\}$  in the next section.

Repeating the previous argument, we conclude after four steps that there are five relevant types of valleys including the three already encountered. Denote by  $T_0, \dots, T_5$  these five types. The first type corresponds to the square configurations  $\eta^{\mathbf{x}}$  and have depth of order  $e^{2\beta}$ . All the others have depth of order  $e^\beta$ . From a valley of type  $T_j$  the process may jump only to a valley of type  $T_i$  for  $0 \leq i \leq 4$ ,  $|i - j| \leq 1$ . Furthermore, on the time scale  $e^\beta$ , the process exhibits a tunneling behavior among all these valleys, the deepest valleys being the absorbing points of the asymptotic Markovian dynamics.

In the next section we present the shallow valleys of type  $T_1, \dots, T_4$  and examine some of their properties. In the following section we prove the tunneling behavior of the process  $\eta_t$  on the time scale  $e^\beta$  starting from one of the valleys of type  $T_j$ ,  $0 \leq j \leq 4$ . The rates of the asymptotic Markovian dynamics are expressed as functions of the transition probabilities of simple random walks.

In Section 6, we deduce from the previous result the tunneling behavior of the process  $\eta_t$  on the longer time scale  $e^{2\beta}$  among the competing metastable states  $\eta^\mathbf{x}$ . The rates of the asymptotic dynamics are expressed in terms of the hitting probabilities of the absorbing states for the Markovian dynamics derived in the previous step.

Since on the time scale  $e^\beta$ , as for a birth and death chain, the process may only jump from a valley of type  $T_j$  to a valley of type  $T_{j+i}$ ,  $|i| \leq 1$ , it should be possible to estimate the dependence on  $n$  and on  $|\mathbf{x} - \mathbf{y}|$ , as  $L \uparrow \infty$ , of the asymptotic jump rates in the time scale  $e^{2\beta}$ .

#### 4. SOME SHALLOW VALLEYS

We examine in this section the evolution of the Markov process  $\{\eta_t^\beta : t \geq 0\}$  between two consecutive visits to the set  $\Omega^0 = \{\eta^\mathbf{x} : \mathbf{x} \in \Lambda_L\}$  of square configurations. In the next section we show that at very low temperatures during these excursions, in a time scale much smaller than the one of the excursions, the process evolves as a continuous time Markov chain whose state space consists of a set of shallow valleys. We present in this section the valleys and the jump rates of this asymptotic chain in terms of two elementary random walks.

Denote by  $H_\Pi$ ,  $H_\Pi^+$ ,  $\Pi \subset \Omega_{L,K}$ , the hitting time and the time of the first return to  $\Pi$ :

$$\begin{aligned} H_\Pi &= \inf \{t > 0 : \eta_t^\beta \in \Pi\}, \\ H_\Pi^+ &= \inf \{t > 0 : \eta_t^\beta \in \Pi \text{ and } \exists 0 < s < t; \eta_s^\beta \notin \Pi\}. \end{aligned}$$

We sometimes write  $H(\Pi)$ ,  $H^+(\Pi)$  instead of  $H_\Pi$ ,  $H_\Pi^+$ .

Let  $\{X_t : t \geq 0\}$  be the nearest-neighbor symmetric random walk on  $\Lambda_L$ . Denote by  $\mathbb{P}_\mathbf{x}$ ,  $\mathbf{x} \in \Lambda_L$ , the probability measure on  $D(\mathbb{R}_+, \Lambda_L)$  induced by  $X_t$  starting from  $\mathbf{x}$ . We sometimes represent  $X_t$  by  $X(t)$ . Denote by  $p(\mathbf{x}, \mathbf{y}, G)$ ,  $\mathbf{x} \in \Lambda_L$ ,  $\mathbf{y} \in G$ ,  $G \subset \Lambda_L$ , the probability that the random walk starting from  $\mathbf{x}$  reaches  $G$  at  $\mathbf{y}$ :

$$p(\mathbf{x}, \mathbf{y}, G) := \mathbb{P}_\mathbf{x}[X(H_G) = \mathbf{y}]. \quad (4.1)$$

Consider two independent, nearest-neighbor, symmetric random walks  $(X_t, Y_t)$  on  $\{0, \dots, n-1\}$ . The jump rates  $r(a, b)$  of each coordinate are given by  $r(0, 1) = r(n-1, n-2) = r(a, a \pm 1) = 1$ ,  $1 \leq a \leq n-2$ . Let  $\mathbb{P}_{(a,b)}^{(1)}$  be the probability on the path space induced by the pair  $(X_t, Y_t)$  starting from  $(a, b)$ . Denote by  $H_a^Y$  the hitting time of  $a$  by the random walk  $Y_t$ :  $H_a^Y = \inf\{t > 0 : Y_t = a\}$  and let  $H^Y = \min\{H_0^Y, H_{n-1}^Y\}$ ,

$$p_1(a) = \mathbb{P}_{(a,n-2)}^{(1)}[X(H^Y) = n-1, H^Y = H_{n-1}^Y].$$

By independence and reversibility,

$$p_1(a) = \mathbb{P}_{(n-1,n-2)}^{(1)}[X(H^Y) = a, H^Y = H_{n-1}^Y], \quad 0 \leq a \leq n-2,$$

so that  $\sum_{0 \leq a \leq n-2} p_1(a) = 1 - p_1(n-1)$ . This last expression will appear below and deserves a special notation. Let

$$p_1 = p_1(n-1) = \mathbb{P}_{(n-1, n-2)}^{(1)}[X(H^Y) = n-1, H^Y = H_{n-1}^Y] . \quad (4.2)$$

Consider the same pair of independent, nearest-neighbor, symmetric random walks  $(X_t, Y_t)$  evolving on an interval  $J = \{u, \dots, v\}$  instead of the interval  $\{0, \dots, n-1\}$ . Let  $H_1 = \inf\{t > 0 : |X_t - Y_t| = 1\}$ . For  $a, a+2 \in J$ ,  $b, b+1 \in J$ , let

$$p(J, a, b) := \mathbb{P}_{(a, a+2)}^{(1)}[(X_{H_1}, Y_{H_1}) = (b, b+1)] .$$

**4.1. The valleys  $\mathcal{E}_x^{i,j}$ .** Let  $Q^i = Q \setminus \{\mathbf{w}_i\}$ ,  $0 \leq i \leq 3$ , where

$$\mathbf{w}_0 = \mathbf{w} = (0, 0) , \quad \mathbf{w}_1 = (n-1, 0) , \quad \mathbf{w}_2 = (n-1, n-1) , \quad \mathbf{w}_3 = (0, n-1)$$

are the corners of the square  $Q$ . For  $\mathbf{x} \in \Lambda_L$ ,  $0 \leq i \leq 3$ , let  $Q_x^i = \mathbf{x} + Q^i$ ,  $\mathbf{x}_i = \mathbf{x} + \mathbf{w}_i$ . For a subset  $\Pi$  of  $\Lambda_L$ , denote by  $\partial\Pi$  the outer boundary of  $\Pi$ . This is the set of sites which are at distance one from  $\Pi$ :  $\partial\Pi = \{\mathbf{x} \in \Lambda_L \setminus \Pi : \exists \mathbf{y} \in \Pi : |\mathbf{y} - \mathbf{x}| = 1\}$ .

Fix  $\mathbf{x} \in \Lambda_L$  and  $0 \leq i \leq 3$ . For  $\mathbf{z} \in \partial Q_x^i \setminus Q_x$ , denote by  $Q_x^{i,\mathbf{z}}$  the set  $Q_x^i \cup \{\mathbf{z}\}$ , by  $\eta^{\mathbf{x}_i, \mathbf{z}} = \sigma^{\mathbf{x}_i, \mathbf{z}} \eta^{\mathbf{x}}$  the corresponding configuration, and by  $\Omega^1 = \Omega_{L,K}^1$  the set of such configurations:

$$\Omega^1 = \{\eta^{\mathbf{x}_i, \mathbf{z}} : \mathbf{x} \in \Lambda_L, 0 \leq i \leq 3, \mathbf{z} \in \partial Q_x^i \setminus Q_x\} .$$

Denote by  $\partial_j Q_x^i$ ,  $0 \leq j \leq 3$ , the  $j$ -th boundary of  $Q_x^i$ :

$$\partial_j Q_x^i = \{\mathbf{z} \in \partial Q_x^i : \exists \mathbf{y} \in Q_x^i : \mathbf{y} - \mathbf{z} = (1-j)e_2\} \quad j = 0, 2 ,$$

$$\partial_j Q_x^i = \{\mathbf{z} \in \partial Q_x^i : \exists \mathbf{y} \in Q_x^i : \mathbf{y} - \mathbf{z} = (j-2)e_1\} \quad j = 1, 3 .$$

Let  $Q_x^{i,j} = \partial_j Q_x^i \setminus Q_x$ ,  $0 \leq i, j \leq 3$ ,  $\mathbf{x} \in \Lambda_L$ , and let  $\mathcal{E}_x^{i,j}$  be the set of configurations in which all sites of the set  $Q_x^i$  are occupied with an extra particle at some location of  $Q_x^{i,j}$ :

$$\mathcal{E}_x^{i,j} = \{\eta^{\mathbf{x}_i, \mathbf{z}} \in \Omega^1 : \mathbf{z} \in Q_x^{i,j}\} .$$

The process  $\{\eta_t^\beta : t \geq 0\}$  can reach any configuration  $\xi \in \mathcal{E}_x^{i,j}$  from any configuration  $\eta \in \mathcal{E}_x^{i,j}$  with rate one jumps. Hence, in the terminology introduced in [4], the sets  $\mathcal{E}_x^{i,j}$  are equivalent classes. Let

$$\Delta_1 = \{\eta \in \Omega_{L,K} : \mu_\beta(\eta) \leq e^{-2\beta} \mu_\beta(\eta^{\mathbf{w}})\} , \quad \Gamma = \Omega_{L,K} \setminus \Delta_1 , \quad (4.3)$$

and note that  $\mathcal{E}_x^{i,j} \subset \Gamma$ . The main result of this subsection states that for any configuration  $\xi \in \mathcal{E}_x^{i,j}$ , the triples  $(\mathcal{E}_x^{i,j}, \mathcal{E}_x^{i,j} \cup \Delta_1, \xi)$  are valleys in the terminology of [2]. In particular, starting from any configuration in  $\mathcal{E}_x^{i,j}$ , the hitting time of the set  $\Gamma \setminus \mathcal{E}_x^{i,j}$  appropriately normalized converges in distribution, as  $\beta \uparrow \infty$ , to an exponential variable. We compute in Proposition 4.2 the time scale which turns the limit a mean one exponential distribution, as well as the asymptotic distribution of  $\eta(H(\Gamma \setminus \mathcal{E}_x^{i,j}))$ .

By symmetry, the distribution of  $\eta(H(\Gamma \setminus \mathcal{E}_x^{i,\cdot}))$  can be obtained from the one of  $\eta(H(\Gamma \setminus \mathcal{E}_x^{0,j}))$ ,  $0 \leq j \leq 3$ . Denote by  $F_x^{i,j}$ , the configurations which do not belong to  $\mathcal{E}_x^{i,j}$ , but which can be reached from a configuration in  $\mathcal{E}_x^{i,j}$  by performing a jump which has rate  $e^{-\beta}$ . The set  $F_{\mathbf{w}}^{2,2}$ , for instance, has the following  $3n$  elements. There are  $n+1$  configurations obtained when the top particle detaches itself from the others:  $\sigma^{\mathbf{w}_2, \mathbf{z}} \eta^{\mathbf{w}}$ , where  $\mathbf{z} = (-1, n)$ ,  $(a, n+1)$ ,  $0 \leq a \leq n-2$ ,  $(n-1, n)$ . There are  $n-1$  configurations obtained when the particle at  $\mathbf{w}_2 - e_2$  moves upward:  $\sigma^{\mathbf{w}_2 - e_2, \mathbf{z}} \eta^{\mathbf{w}}$ ,  $\mathbf{z} = (a, n)$ ,  $0 \leq a \leq n-2$ . There are  $n-2$  configurations obtained when



the particle at  $\mathbf{w}_2 - e_1$  moves to the right:  $\sigma^{\mathbf{w}_2 - e_1, \mathbf{z}} \eta^{\mathbf{w}}$ ,  $\mathbf{z} = (a, n)$ ,  $0 \leq a \leq n - 3$ . To complete the description of the set  $F_{\mathbf{w}}^{2,2}$ , we have to add the configurations  $\sigma^{\mathbf{w}_3, \mathbf{w}_3 + e_2} \sigma^{\mathbf{w}_2, \mathbf{w}_3 + e_1 + e_2} \eta^{\mathbf{w}}$  and  $\sigma^{\mathbf{w}_2 - e_1, \mathbf{w}_2 - e_1 + e_2} \sigma^{\mathbf{w}_2, \mathbf{w}_2 - 2e_1 + e_2} \eta^{\mathbf{w}}$ .

**Lemma 4.1.** *For  $\mathbf{x} \in \Lambda_L$ ,  $0 \leq i, j \leq 3$ , and  $\xi \in F_{\mathbf{x}}^{i,j}$ , there exists a probability measure  $A_{\mathbf{x}}^{i,j}(\xi, \cdot)$  defined on  $\Gamma$  such that*

$$\lim_{\beta \rightarrow \infty} \mathbf{P}_{\xi}^{\beta}[\eta(H_{\Gamma}) \in \Pi] = A_{\mathbf{x}}^{i,j}(\xi, \Pi), \quad \Pi \subset \Gamma.$$

Moreover, in the case  $\mathbf{x} = \mathbf{w}$ ,  $i = j = 2$ ,

$$A_{\mathbf{w}}^{2,2}(\sigma^{\mathbf{w}_2, \mathbf{z}} \eta^{\mathbf{w}}, \Pi) = \begin{cases} p(\mathbf{z}, \mathbf{w}_2, \partial Q^2) & \text{if } \Pi = \{\eta^{\mathbf{w}}\} \\ p(\mathbf{z}, Q_{\mathbf{w}}^{2,j}, \partial Q^2) & \text{if } \Pi = \mathcal{E}_{\mathbf{w}}^{2,j} \end{cases} \quad \mathbf{z} \in J_1, \quad 0 \leq j \leq 3,$$

where  $J_1 = \{(-1, n), (n-1, n), (a, n+1) : 0 \leq a \leq n-2\}$ .

$$A_{\mathbf{w}}^{2,2}(\sigma^{\mathbf{w}_2 - e_2, \mathbf{z}} \eta^{\mathbf{w}}, \Pi) = \begin{cases} \frac{1}{n-1} & \text{if } \Pi = \mathcal{E}_{\mathbf{w}}^{1,2} \\ \frac{n-2}{n-1} - p_1(z_1) [1 - p(\mathbf{w}_2 + e_2, Q_{\mathbf{w}}^{2,2}, \partial Q^2)] & \text{if } \Pi = \mathcal{E}_{\mathbf{w}}^{2,2} \\ p_1(z_1) p(\mathbf{w}_2 + e_2, \mathbf{w}_2, \partial Q^2) & \text{if } \Pi = \{\eta^{\mathbf{w}}\} \\ p_1(z_1) p(\mathbf{w}_2 + e_2, Q_{\mathbf{w}}^{2,j}, \partial Q^2) & \text{if } \Pi = \mathcal{E}_{\mathbf{w}}^{2,j} \end{cases}$$

for  $j \neq 2$ ,  $\mathbf{z} \in Q_{\mathbf{w}}^{2,2}$ , where  $z_1 = \mathbf{z} \cdot e_1$ .

$$A_{\mathbf{w}}^{2,2}(\sigma^{\mathbf{w}_2 - e_1, \mathbf{z}} \eta^{\mathbf{w}}, \mathcal{E}_{\mathbf{w}}^{2,2}) = 1$$

for  $\mathbf{z} = (a, n)$ ,  $0 \leq a \leq n-3$ .

$$A_{\mathbf{w}}^{2,2}(\sigma^{\mathbf{w}_3, \mathbf{w}_3 + e_2} \sigma^{\mathbf{w}_2, \mathbf{w}_3 + e_1 + e_2} \eta^{\mathbf{w}}, \Pi) = \begin{cases} \frac{1}{n} & \text{if } \Pi = \{\sigma^{\mathbf{w}_2, \mathbf{w}_3 + e_1 + e_2} \sigma^{\mathbf{w}_0, \mathbf{w}_3 + e_2} \eta^{\mathbf{w}}\} \\ \frac{n-1}{n} & \text{if } \Pi = \mathcal{E}_{\mathbf{w}}^{2,2}. \end{cases}$$

$$A_{\mathbf{w}}^{2,2}(\sigma^{\mathbf{w}_2 - e_1, \mathbf{w}_2 - e_1 + e_2} \sigma^{\mathbf{w}_2, \mathbf{w}_2 - 2e_1 + e_2} \eta^{\mathbf{w}}, \mathcal{E}_{\mathbf{w}}^{2,2}) = 1.$$

*Proof.* We present the proof for  $i = j = 2$ ,  $\mathbf{x} = \mathbf{w}$ , the other cases being analogous. As we have seen, the set  $F_{\mathbf{w}}^{2,2}$  has five different types of configurations. We examine each one separately. Assume first that  $\xi = \sigma^{\mathbf{w}_2, \mathbf{z}} \eta^{\mathbf{w}}$  for some  $\mathbf{z} \in J_1$ . The free particle, initially at  $\mathbf{z}$  performs a rate one, nearest-neighbor, symmetric random walk in  $\Lambda_L$  until it reaches the boundary of the set  $Q^2$ . All the other possible jumps have rate at most  $e^{-\beta}$  and may therefore be neglected in the argument. When the free particle attains  $\partial Q^2$ , the configuration is  $\eta^{\mathbf{w}}$  with probability  $p(\mathbf{z}, \mathbf{w}_2, \partial Q^2)$ , and it belongs to  $\mathcal{E}_{\mathbf{w}}^{2,j}$  with probability  $p(\mathbf{z}, Q_{\mathbf{w}}^{2,j}, \partial Q^2)$ ,  $0 \leq j \leq 3$ .

Assume now that  $\xi = \sigma^{\mathbf{w}_2 - e_2, \mathbf{z}} \eta^{\mathbf{w}}$ ,  $\mathbf{z} \in Q_{\mathbf{w}}^{2,2}$ . The top particle, initially at  $\mathbf{z}$ , performs a horizontal, symmetric, rate one random walk on the interval  $\{0, \dots, n-1\}$ , while the hole at  $\mathbf{w}_2 - e_2$  performs a vertical, symmetric, rate one random walk on the interval  $\{0, \dots, n-1\}$ . Note that when the top particle is at  $(n-2, n)$  and the hole at  $(n-1, n-2)$ , the particle at  $\mathbf{w}_2$  may jump to  $\mathbf{w}_2 + e_2$  and freeze the top particle. However, this jump may be neglected in the analysis because when the system reaches this configuration the only rate one jump is the return of the particle at  $\mathbf{w}_2 + e_2$  to its original position  $\mathbf{w}_2$ .

The coupled random walk formed by the top particle and the hole evolves until the hole reaches the bottom position  $\mathbf{w}_1$  or it reaches its original position at  $\mathbf{w}_2$ .

At this time, there are three cases. The hole attains  $\mathbf{w}_1$  before  $\mathbf{w}_2$  with probability  $(n-1)^{-1}$ , at which time the configuration belongs to the equivalent class  $\mathcal{E}_{\mathbf{w}}^{1,2}$ .

Recall the definition of  $p_1(\cdot)$  introduced just before (4.2) and let  $z_1 = \mathbf{z} \cdot e_1$ . With probability  $[(n-2)/(n-1)] - p_1(z_1)$  the hole reaches  $\mathbf{w}_2$  before  $\mathbf{w}_1$  at a time where the top particle is not at  $\mathbf{w}_2 + e_2$ . In this case, the process returned to a configuration in the equivalent class  $\mathcal{E}_{\mathbf{w}_2}^{2,2}$ .

With probability  $p_1(z_1)$  the hole reaches  $\mathbf{w}_2$  before  $\mathbf{w}_1$  at a time where the top particle is at  $\mathbf{w}_2 + e_2$ . The configuration at this time is  $\sigma^{\mathbf{w}_2, \mathbf{w}_2 + e_2} \eta^{\mathbf{w}}$ . From this point, the detached particle at  $\mathbf{w}_2 + e_2$  performs a rate one random walk in  $\Lambda_L$  until it reaches to boundary of  $Q^2$ . At this hitting time, the configuration is  $\eta^{\mathbf{w}}$  with probability  $p(\mathbf{w}_2 + e_2, \mathbf{w}_2, \partial Q^2)$ , and it belongs to  $\mathcal{E}_{\mathbf{w}}^{2,j}$  with probability  $p(\mathbf{w}_2 + e_2, Q_{\mathbf{w}}^{2,j}, \partial Q^2)$ ,  $0 \leq j \leq 3$ .

In the case  $\xi = \sigma^{\mathbf{w}_2 - e_1, \mathbf{z}} \eta^{\mathbf{w}}$ , the hole at  $\mathbf{w}_2 - e_1$  performs a horizontal, rate one, symmetric random walk on the interval  $\{m+1, \dots, n-1\}$ , where  $m$  represents the horizontal position of the top particle, which itself performs a horizontal, rate one, symmetric random walk limited on its right by the hole in the row below. This coupled system evolves until the hole initially at  $\mathbf{w}_2 - e_1$  reaches its original position at  $\mathbf{w}_2$ .

Suppose that  $\xi = \sigma^{\mathbf{w}_3, \mathbf{w}_3 + e_2} \sigma^{\mathbf{w}_2, \mathbf{w}_3 + e_1 + e_2} \eta^{\mathbf{w}}$ . In this situation the hole at  $\mathbf{w}_3$  performs a vertical, rate one, symmetric random walk on  $\{(0, b) : 0 \leq b \leq n\}$ . The hole reaches  $\mathbf{w}$  before it reaches  $\mathbf{w}_3$  with probability  $n^{-1}$ .

Finally, if  $\xi = \sigma^{\mathbf{w}_2 - e_1, \mathbf{w}_2 - e_1 + e_2} \sigma^{\mathbf{w}_2, \mathbf{w}_2 - 2e_1 + e_2} \eta^{\mathbf{w}}$ , there is only one rate one jump which drives the system back to the set  $\mathcal{E}_{\mathbf{w}}^{2,2}$ .  $\square$

When the set  $\Pi$  is a singleton  $\{\zeta\}$  we represent  $A_{\mathbf{x}}^{i,j}(\xi, \{\zeta\})$  by  $A_{\mathbf{x}}^{i,j}(\xi, \zeta)$ . This convention is adopted for all functions of sets without further comment. Recall the definition of valley presented in [2], and the notation introduced in the beginning of this section and in the statement of Lemma 4.1. Let

$$Z(\mathcal{E}_{\mathbf{x}}^{i,j}) = \sum_{\xi \in F_{\mathbf{x}}^{i,j}} A_{\mathbf{x}}^{i,j}(\xi, (\mathcal{E}_{\mathbf{x}}^{i,j})^c) = \sum_{\xi \in F_{\mathbf{x}}^{i,j}} \{1 - A_{\mathbf{x}}^{i,j}(\xi, \mathcal{E}_{\mathbf{x}}^{i,j})\}.$$

Note that  $Z(\mathcal{E}_{\mathbf{x}}^{i,j})$  does not depend on  $\mathbf{x}$ . In view of the previous lemma,

$$Z(\mathcal{E}_{\mathbf{x}}^{2,2}) = 1 + \frac{1}{n} + [1 - p_1] [1 - p(\mathbf{w}_2 + e_2, Q_{\mathbf{w}}^{2,2}, \partial Q^2)] + \sum_{\mathbf{z} \in J_1} [1 - p(\mathbf{z}, Q_{\mathbf{w}}^{2,2}, \partial Q^2)].$$

**Proposition 4.2.** Fix  $0 \leq i, j \leq 3$  and  $\mathbf{x} \in \Lambda_L$ .

- (1) For any  $\xi \in \mathcal{E}_{\mathbf{x}}^{i,j}$ , the triple  $(\mathcal{E}_{\mathbf{x}}^{i,j}, \mathcal{E}_{\mathbf{x}}^{i,j} \cup \Delta_1, \xi)$  is a valley of depth  $\mu_{\beta}(\mathcal{E}_{\mathbf{x}}^{i,j}) / \text{cap}_{\beta}(\mathcal{E}_{\mathbf{x}}^{i,j}, [\mathcal{E}_{\mathbf{x}}^{i,j} \cup \Delta_1]^c)$ ;
- (2) For any  $\xi \in \mathcal{E}_{\mathbf{x}}^{i,j}$ , under  $\mathbf{P}_{\xi}^{\beta}$ ,  $H(\Gamma \setminus \mathcal{E}_{\mathbf{x}}^{i,j}) / e^{\beta}$  converges in distribution to an exponential random variable of parameter  $Z(\mathcal{E}_{\mathbf{x}}^{i,j})$ ;
- (3) For any  $\xi \in \mathcal{E}_{\mathbf{x}}^{i,j}$ ,  $\Pi \subset \Gamma \setminus \mathcal{E}_{\mathbf{x}}^{i,j}$ ,

$$\lim_{\beta \rightarrow \infty} \mathbf{P}_{\xi}^{\beta} [\eta(H(\Gamma \setminus \mathcal{E}_{\mathbf{x}}^{i,j})) \in \Pi] = \frac{1}{Z(\mathcal{E}_{\mathbf{x}}^{i,j})} \sum_{\eta \in F_{\mathbf{x}}^{i,j}} A_{\mathbf{x}}^{i,j}(\eta, \Pi) =: Q(\mathcal{E}_{\mathbf{x}}^{i,j}, \Pi).$$

*Proof.* Recall [2, Theorem 2.6]. Condition (2.15) is fulfilled by definition of the set  $\Delta_1$ . A simple argument shows that  $G_{\beta}(\xi, \zeta) = e^{-\beta} \mu_{\beta}(\eta^{\mathbf{w}})$  for any pair of configurations  $\xi \neq \zeta \in \mathcal{E}_{\mathbf{x}}^{i,j}$ , and that  $G_{\beta}(\mathcal{E}_{\mathbf{x}}^{i,j}, [\mathcal{E}_{\mathbf{x}}^{i,j} \cup \Delta_1]^c) \leq e^{-2\beta} \mu_{\beta}(\eta^{\mathbf{w}})$ . Condition

(2.14) follows from these estimates and (3.1). This proves the first assertion of the lemma.

To prove the second assertion of the lemma, we start with a recursive formula for  $H_\Gamma^\pm$ . Let  $\tau_1$  the time the process leaves the set  $\mathcal{E}_\mathbf{x}^{i,j}$ :  $\tau_1 = \inf\{t > 0 : \eta_t^\beta \notin \mathcal{E}_\mathbf{x}^{i,j}\}$ . We have that

$$H_{\Gamma \setminus \mathcal{E}_\mathbf{x}^{i,j}} = \tau_1 + H_\Gamma \circ \theta_{\tau_1} + \mathbf{1}\{H_\Gamma \circ \theta_{\tau_1} = H_{\mathcal{E}_\mathbf{x}^{i,j}} \circ \theta_{\tau_1}\} H_{\Gamma \setminus \mathcal{E}_\mathbf{x}^{i,j}} \circ \theta_{H_\Gamma^\pm},$$

where  $\{\theta_t : t \geq 0\}$  stands for the shift operators.

Fix  $\lambda > 0$  and let  $\lambda_\beta = \lambda e^{-\beta}$ . By the strong Markov property, for any  $\xi \in \mathcal{E}_\mathbf{x}^{i,j}$ ,

$$\begin{aligned} \mathbf{E}_\xi^\beta \left[ e^{-\lambda_\beta H_{\Gamma \setminus \mathcal{E}_\mathbf{x}^{i,j}}} \right] &= \mathbf{E}_\xi^\beta \left[ e^{-\lambda_\beta \tau_1} \mathbf{E}_{\eta_{\tau_1}}^\beta \left[ \mathbf{1}\{H_\Gamma \neq H_{\mathcal{E}_\mathbf{x}^{i,j}}\} e^{-\lambda_\beta H_\Gamma} \right] \right] \\ &\quad + \mathbf{E}_\xi^\beta \left[ e^{-\lambda_\beta \tau_1} \mathbf{E}_{\eta_{\tau_1}}^\beta \left[ \mathbf{1}\{H_\Gamma = H_{\mathcal{E}_\mathbf{x}^{i,j}}\} e^{-\lambda_\beta H_\Gamma} \exp \left\{ -\lambda_\beta H_{\Gamma \setminus \mathcal{E}_\mathbf{x}^{i,j}} \circ \theta_{H_\Gamma} \right\} \right] \right]. \end{aligned} \quad (4.4)$$

Recall the definition of  $F_\mathbf{x}^{i,j}$  given just before the statement of Lemma 4.1. With a probability which converges to 1 as  $\beta \uparrow \infty$ ,  $\eta_{\tau_1}^\beta$  belongs to  $F_\mathbf{x}^{i,j}$ . Each configuration in  $F_\mathbf{x}^{i,j}$  belongs to an equivalent class which eventually attains  $\Gamma$  after a finite random number of rate one jumps. This proves that

$$\lim_{A \rightarrow \infty} \lim_{\beta \rightarrow \infty} \max_{\zeta \in F_\mathbf{x}^{i,j}} \mathbf{P}_\zeta^\beta [H_\Gamma > A] = 0.$$

Therefore, we may replace in (4.4)  $\exp\{-\lambda_\beta H_\Gamma\}$  by 1 at a cost which vanishes as  $\beta \uparrow \infty$ .

By the strong Markov property, after the last replacement, the second term on the right hand side of (4.4) can be rewritten as

$$\mathbf{E}_\xi^\beta \left[ e^{-\lambda_\beta \tau_1} \mathbf{E}_{\eta_{\tau_1}}^\beta \left[ \mathbf{1}\{H_\Gamma = H_{\mathcal{E}_\mathbf{x}^{i,j}}\} \mathbf{E}_{\eta_{H_\Gamma}}^\beta \left[ \exp \left\{ -\lambda_\beta H_{\Gamma \setminus \mathcal{E}_\mathbf{x}^{i,j}} \right\} \right] \right] \right].$$

Since  $\mathcal{E}_\mathbf{x}^{i,j}$  is an equivalent class and the process leaves  $\mathcal{E}_\mathbf{x}^{i,j}$  only after a rate  $e^{-\beta}$  jump, a simple coupling argument shows that

$$\lim_{\beta \rightarrow \infty} \max_{\eta, \zeta \in \mathcal{E}_\mathbf{x}^{i,j}} \left| \mathbf{E}_\eta^\beta \left[ \exp \left\{ -\lambda_\beta H_{\Gamma \setminus \mathcal{E}_\mathbf{x}^{i,j}} \right\} \right] - \mathbf{E}_\zeta^\beta \left[ \exp \left\{ -\lambda_\beta H_{\Gamma \setminus \mathcal{E}_\mathbf{x}^{i,j}} \right\} \right] \right| = 0.$$

The previous expectation is thus equal to

$$\mathbf{E}_\xi^\beta \left[ \exp \left\{ -\lambda_\beta H_{\Gamma \setminus \mathcal{E}_\mathbf{x}^{i,j}} \right\} \right] \mathbf{E}_\xi^\beta \left[ e^{-\lambda_\beta \tau_1} \mathbf{E}_{\eta_{\tau_1}}^\beta \left[ \mathbf{1}\{H_\Gamma = H_{\mathcal{E}_\mathbf{x}^{i,j}}\} \right] \right]$$

plus an error which vanishes as  $\beta \uparrow \infty$ .

We claim that  $(e^{-\beta} \tau_1, \eta_{\tau_1}^\beta)$  converges in distribution, as  $\beta \uparrow \infty$ , to a pair of independent random variables where the first coordinate is an exponential time and the second coordinate has a distribution concentrated on  $F_\mathbf{x}^{i,j}$ . The proof of this claim relies on [2, Theorem 2.7] and on a coupling argument.

Let  $G_\mathbf{x}^{i,j} = F_\mathbf{x}^{i,j} \cup \mathcal{E}_\mathbf{x}^{i,j}$ . Consider the Markov process  $\{\hat{\eta}_t^\beta : t \geq 0\}$  on  $G_\mathbf{x}^{i,j}$  whose jump rates  $\hat{r}(\eta, \xi)$  are given by

$$\hat{r}(\eta, \xi) = \begin{cases} r(\eta, \xi) & \text{if } \eta \in \mathcal{E}_\mathbf{x}^{i,j}, \xi \in G_\mathbf{x}^{i,j}, \\ r(\xi, \eta) & \text{if } \eta \in F_\mathbf{x}^{i,j}, \xi \in \mathcal{E}_\mathbf{x}^{i,j}, \\ 0 & \text{otherwise.} \end{cases}$$

Note that  $\hat{r}(\eta, \xi) = e^{-\beta}$  or 0 if  $\eta \in F_\mathbf{x}^{i,j}$ , and that we may couple the processes  $\eta_t^\beta$  and  $\hat{\eta}_t^\beta$  in such a way that the probability of the event  $\{\eta_t^\beta = \hat{\eta}_t^\beta : 0 \leq t \leq \tau_1\}$  converges to one as  $\beta \uparrow \infty$  if the initial state belongs to  $\mathcal{E}_\mathbf{x}^{i,j}$ .

Let  $\{\xi^1, \dots, \xi^m\}$  be an enumeration of the set  $F_{\mathbf{x}}^{i,j}$  and consider the partition  $\mathcal{E}_{\mathbf{x}}^{i,j} \cup \{\xi^1\} \cup \dots \cup \{\xi^m\}$  of the set  $G_{\mathbf{x}}^{i,j}$ . Assumption (H1) of [2, Theorem 2.7] for the process  $\hat{\eta}_t^\beta$  is empty for the sets  $\{\xi^j\}$  and has been checked in the first part of this proof for the set  $\mathcal{E}_{\mathbf{x}}^{i,j}$ . Assumption (H0) for the process  $\hat{\eta}_t^\beta$  speeded up by  $e^\beta$  can be verified by a direct computation. Therefore, by [2, Theorem 2.7], the pair  $(e^{-\beta}\hat{\tau}_1, \hat{\eta}_{\tau_1}^\beta)$  converges in distribution, as  $\beta \uparrow \infty$ , to a pair of independent random variables in which the first coordinate has an exponential distribution and the second one is concentrated over  $F_{\mathbf{x}}^{i,j}$ . This result can be extended to the original pair  $(e^{-\beta}\tau_1, \eta_{\tau_1}^\beta)$  by the coupling argument alluded to above.

It follows from the claim just proved and the previous estimates that

$$\lim_{\beta \rightarrow \infty} \mathbf{E}_\xi^\beta \left[ e^{-\lambda_\beta H_{\Gamma \setminus \mathcal{E}_{\mathbf{x}}^{i,j}}} \right] = \lim_{\beta \rightarrow \infty} \frac{\mathbf{E}_\xi^\beta [e^{-\lambda_\beta \tau_1}] \mathbf{E}_\xi^\beta [\mathbf{P}_{\eta_{\tau_1}}^\beta [H_\Gamma \neq H_{\mathcal{E}_{\mathbf{x}}^{i,j}}]]}{1 - \mathbf{E}_\xi^\beta [e^{-\lambda_\beta \tau_1}] \mathbf{E}_\xi^\beta [\mathbf{P}_{\eta_{\tau_1}}^\beta [H_\Gamma = H_{\mathcal{E}_{\mathbf{x}}^{i,j}}]]}.$$

If  $\tau_1/e^\beta$  converges to an exponential random variable of parameter  $\theta$ , the right hand side becomes

$$\lim_{\beta \rightarrow \infty} \frac{\theta \mathbf{E}_\xi^\beta [\mathbf{P}_{\eta_{\tau_1}}^\beta [H_\Gamma \neq H_{\mathcal{E}_{\mathbf{x}}^{i,j}}]]}{\lambda + \theta \mathbf{E}_\xi^\beta [\mathbf{P}_{\eta_{\tau_1}}^\beta [H_\Gamma \neq H_{\mathcal{E}_{\mathbf{x}}^{i,j}}]]},$$

which means that  $H(\Gamma \setminus \mathcal{E}_{\mathbf{x}}^{i,j})/e^\beta$  converges to an exponential random variable of parameter

$$\gamma = \theta \lim_{\beta \rightarrow \infty} \mathbf{E}_\xi^\beta [\mathbf{P}_{\eta_{\tau_1}}^\beta [H_\Gamma \neq H_{\mathcal{E}_{\mathbf{x}}^{i,j}}]].$$

We examine the case  $i = j = 2$ ,  $\mathbf{x} = \mathbf{w}$ . Recall the description of the set  $F_{\mathbf{w}}^{2,2}$  presented before Lemma 4.1. By computing the average rates which appear in assumption (H0) of [2], we obtain that under  $\mathbf{P}_\xi^\beta$ ,  $\tau_1/e^\beta$  converges in distribution to an exponential random variable of parameter  $3n$ , and that  $\eta_{\tau_1}^\beta$  converges to a uniform distribution on  $F_{\mathbf{w}}^{2,2}$ . Hence, since  $F_{\mathbf{w}}^{2,2}$  has  $3n$  elements, by the conclusions of the previous paragraph and by Lemma 4.1,  $H(\Gamma \setminus \mathcal{E}_{\mathbf{x}}^{i,j})/e^\beta$  converges to an exponential random variable of parameter  $Z(\mathcal{E}_{\mathbf{x}}^{i,j})$ . This proves the second assertion of the proposition.

We turn to the third assertion. Denote by  $\{H_j : j \geq 1\}$  the successive return times to  $\Gamma$ :

$$H_1 = H^+(\Gamma), \quad H_{j+1} = H^+(\Gamma) \circ \theta_{H_j}, \quad j \geq 1.$$

With this notation, we may write for every  $\xi \in \mathcal{E}_{\mathbf{w}}^{2,2}$ ,

$$\mathbf{P}_\xi^\beta [\eta(H_{\Gamma \setminus \mathcal{E}_{\mathbf{w}}^{2,2}}) \in \Pi] = \sum_{j \geq 1} \mathbf{P}_\xi^\beta [\eta(H_k) \in \mathcal{E}_{\mathbf{w}}^{2,2}, 1 \leq k \leq j-1, \eta(H_j) \in \Pi]. \quad (4.5)$$

By the strong Markov property, for any  $\xi' \in \mathcal{E}_{\mathbf{w}}^{2,2}$ ,  $\Pi' \subset \Gamma$ ,

$$\mathbf{P}_{\xi'}^\beta [\eta(H_1) \in \Pi'] = \mathbf{E}_{\xi'}^\beta [\mathbf{P}_{\eta_{\tau_1}}^\beta [\eta(H_\Gamma) \in \Pi']].$$

Under  $\mathbf{P}_\xi^\beta$ , the distribution of  $\eta_{\tau_1}$  converges to the uniform distribution over  $F_{\mathbf{w}}^{2,2}$  as  $\beta \uparrow \infty$ . Hence, by Lemma 4.1,

$$\lim_{\beta \rightarrow \infty} \mathbf{P}_{\xi'}^\beta [\eta(H_1) \in \Pi'] = \frac{1}{|F_{\mathbf{w}}^{2,2}|} \sum_{\eta \in F_{\mathbf{w}}^{2,2}} A_{\mathbf{w}}^{2,2}(\eta, \Pi')$$

for all  $\xi' \in \mathcal{E}_{\mathbf{w}}^{2,2}$ ,  $\Pi' \subset \Gamma$ . Denote the right hand side of the previous formula by  $q(\Pi')$ . It follows from identity (4.5), the strong Markov property and the previous observation that for all  $\Pi \subset \Gamma \setminus \mathcal{E}_{\mathbf{w}}^{2,2}$ ,

$$\lim_{\beta \rightarrow \infty} \mathbf{P}_{\xi}^{\beta} [\eta(H_{\Gamma \setminus \mathcal{E}_{\mathbf{w}}^{2,2}}) \in \Pi] = \frac{q(\Pi)}{1 - q(\mathcal{E}_{\mathbf{w}}^{2,2})},$$

which concludes the proof of the proposition.  $\square$

Since  $\Gamma \setminus \mathcal{E}_{\mathbf{x}}^{i,j} = \mathcal{E}_{\mathbf{x}}^{i,j} \cup \Delta_1$ , it follows from the second assertion of the lemma that the depth of the valley  $(\mathcal{E}_{\mathbf{x}}^{i,j}, \mathcal{E}_{\mathbf{x}}^{i,j} \cup \Delta_1, \xi)$  is  $e^{\beta}/Z(\mathcal{E}_{\mathbf{x}}^{i,j})$ . In particular,

$$\lim_{\beta \rightarrow \infty} \frac{\mu_{\beta}(\mathcal{E}_{\mathbf{x}}^{i,j})}{e^{\beta} \text{cap}_{\beta}(\mathcal{E}_{\mathbf{x}}^{i,j}, [\mathcal{E}_{\mathbf{x}}^{i,j} \cup \Delta_1]^c)} = \frac{1}{Z(\mathcal{E}_{\mathbf{x}}^{i,j})}. \quad (4.6)$$

Since  $\mu_{\beta}(\mathcal{E}_{\mathbf{x}}^{i,j}) = |\mathcal{E}_{\mathbf{x}}^{i,j}| e^{-\beta} \mu_{\beta}(\eta^{\mathbf{w}})$ , where  $|A|$  represents the cardinality of  $A$ ,

$$\lim_{\beta \rightarrow \infty} \frac{\text{cap}_{\beta}(\mathcal{E}_{\mathbf{x}}^{i,j}, [\mathcal{E}_{\mathbf{x}}^{i,j} \cup \Delta_1]^c)}{e^{-2\beta} \mu_{\beta}(\eta^{\mathbf{w}})} = |\mathcal{E}_{\mathbf{x}}^{i,j}| Z(\mathcal{E}_{\mathbf{x}}^{i,j}).$$

We have the following explicit formula for the probability measure  $Q(\mathcal{E}_{\mathbf{w}}^{2,2}, \cdot)$  on  $\Gamma$ .  $Q(\mathcal{E}_{\mathbf{w}}^{2,2}, \Pi)$  is equal to

$$\frac{1}{Z(\mathcal{E}_{\mathbf{w}}^{2,2})} \begin{cases} \sum_{\mathbf{z} \in J_1} p(\mathbf{z}, \mathbf{w}_2, \partial Q^2) + [1 - p_1] p(\mathbf{w}_2 + e_2, \mathbf{w}_2, \partial Q^2) & \text{if } \Pi = \{\eta^{\mathbf{w}}\} \\ \frac{1}{n} & \text{if } \Pi = \{\sigma^{\mathbf{w}_2, \mathbf{w}_3 + e_1 + e_2} \sigma^{\mathbf{w}_0, \mathbf{w}_3 + e_2} \eta^{\mathbf{w}}\} \\ \sum_{\mathbf{z} \in J_1} p(\mathbf{z}, Q_{\mathbf{w}}^{2,j}, \partial Q^2) + [1 - p_1] p(\mathbf{w}_2 + e_2, Q_{\mathbf{w}}^{2,j}, \partial Q^2) & \text{if } \Pi = \mathcal{E}_{\mathbf{w}}^{2,j} \\ 1 & \text{if } \Pi = \mathcal{E}_{\mathbf{w}}^{1,2} \end{cases}$$

for  $j \neq 2$ . Hence, on the time scale  $e^{\beta}$ , starting from the valley  $\mathcal{E}_{\mathbf{w}}^{2,2}$  the process may fall in the deep well  $\eta^{\mathbf{w}}$ , it may reach the valleys  $\mathcal{E}_{\mathbf{w}}^{1,2}$ ,  $\mathcal{E}_{\mathbf{w}}^{2,j}$ , which are similar to  $\mathcal{E}_{\mathbf{w}}^{2,2}$ , or attain the valley formed by the single configuration  $\sigma^{\mathbf{w}_2, \mathbf{w}_3 + e_1 + e_2} \sigma^{\mathbf{w}_0, \mathbf{w}_3 + e_2} \eta^{\mathbf{w}}$ . This new type of valley is studied in the next subsection.

**4.2. The valleys  $\{\eta_{\mathbf{x}}^{\mathbf{a}, (\mathbf{k}, \ell)}\}$ .** Let  $R^{\mathbf{l}}, R^{\mathbf{s}}$  be the rectangles  $R^{\mathbf{l}} = \{1, \dots, n-1\} \times \{1, \dots, n-2\}$ ,  $R^{\mathbf{s}} = \{1, \dots, n-2\} \times \{1, \dots, n-1\}$ , where  $\mathbf{l}$  stands for lying and  $\mathbf{s}$  for standing. Let  $n_0^{\mathbf{s}} = n_2^{\mathbf{s}} = n-2$ ,  $n_1^{\mathbf{s}} = n_3^{\mathbf{s}} = n-1$  be the length of the sides of the standing rectangle  $R^{\mathbf{s}}$ . Similarly, denote by  $n_i^{\mathbf{l}}$ ,  $0 \leq i \leq 3$ , the length of the sides of the lying rectangle  $R^{\mathbf{l}}$ :  $n_i^{\mathbf{l}} = n_{i+1}^{\mathbf{s}}$ , where the sum over the index  $i$  is performed modulo 4.

Denote by  $\mathbb{I}_{\mathbf{a}}$ ,  $\mathbf{a} \in \{\mathbf{s}, \mathbf{l}\}$ , the set of pairs  $(\mathbf{k}, \ell) = (k_0, \ell_0; k_1, \ell_1; k_2, \ell_2; k_3, \ell_3)$  such that

- $0 \leq k_i \leq \ell_i \leq n_i^{\mathbf{a}}$ ,
- If  $k_j = 0$ , then  $\ell_{j-1} = n_{j-1}^{\mathbf{a}}$ .

For  $(\mathbf{k}, \ell) \in \mathbb{I}_{\mathbf{a}}$ ,  $\mathbf{a} \in \{\mathbf{s}, \mathbf{l}\}$ , let  $R^{\mathbf{l}}(\mathbf{k}, \ell)$ ,  $R^{\mathbf{s}}(\mathbf{k}, \ell)$  be the sets

$$\begin{aligned} R^{\mathbf{l}}(\mathbf{k}, \ell) &= R^{\mathbf{l}} \cup \{(a, 0) : k_0 \leq a \leq \ell_0\} \cup \{(n, b) : k_1 \leq b \leq \ell_1\} \cup \\ &\quad \cup \{(n-a, n-1) : k_2 \leq a \leq \ell_2\} \cup \{(0, n-1-b) : k_3 \leq b \leq \ell_3\}, \\ R^{\mathbf{s}}(\mathbf{k}, \ell) &= R^{\mathbf{s}} \cup \{(a, 0) : k_0 \leq a \leq \ell_0\} \cup \{(n-1, b) : k_1 \leq b \leq \ell_1\} \cup \\ &\quad \cup \{(n-1-a, n) : k_2 \leq a \leq \ell_2\} \cup \{(0, n-b) : k_3 \leq b \leq \ell_3\}. \end{aligned}$$

Note that a hole between particles on the side of a rectangle is not allowed in the sets  $R^a(\mathbf{k}, \ell)$ ,  $R^s(\mathbf{k}, \ell)$ .

Denote by  $I_a$ ,  $a \in \{\mathfrak{s}, \mathfrak{l}\}$ , the set of pairs  $(\mathbf{k}, \ell) \in \mathbb{I}_a$  such that  $|R^a(\mathbf{k}, \ell)| = n^2$ . For  $(\mathbf{k}, \ell) \in I_a$ , denote by  $M_i(\mathbf{k}, \ell)$  the number of particles attached to the side  $i$  of the rectangle  $R^a(\mathbf{k}, \ell)$ :

$$M_i(\mathbf{k}, \ell) = \begin{cases} \ell_i - k_i + 1 & \text{if } k_{i+1} \geq 1, \\ \ell_i - k_i + 2 & \text{if } k_{i+1} = 0. \end{cases}$$

Clearly, for  $(\mathbf{k}, \ell) \in I_a$ ,  $\sum_{0 \leq i \leq 3} M_i(\mathbf{k}, \ell) = 3n - 2 + A$ , where  $A$  is the number of occupied corners, which are counted twice since they are attached to two sides.

Denote by  $I_a^* \subset I_a$ , the set of pairs  $(\mathbf{k}, \ell) \in I_a$  whose rectangles  $R^a(\mathbf{k}, \ell)$  have at least two particles on each side:  $M_i(\mathbf{k}, \ell) \geq 2$ ,  $0 \leq i \leq 3$ . Note that if  $(\mathbf{k}, \ell)$  belongs to  $I_a^*$ , for all  $\mathbf{x} \in R^a(\mathbf{k}, \ell)$ , there exist  $\mathbf{y}, \mathbf{z} \in R^a(\mathbf{k}, \ell)$ ,  $\mathbf{y} \neq \mathbf{z}$ , with the property  $|\mathbf{x} - \mathbf{y}| = |\mathbf{x} - \mathbf{z}| = 1$ .

For  $(\mathbf{k}, \ell) \in I_a$ ,  $a \in \{\mathfrak{s}, \mathfrak{l}\}$ ,  $\mathbf{x} \in \Lambda_L$ , let  $R_{\mathbf{x}}^a(\mathbf{k}, \ell) = \mathbf{x} + R^a(\mathbf{k}, \ell)$ , and let  $\eta_{\mathbf{x}}^{a, (\mathbf{k}, \ell)}$  represent the configurations defined by

$$\eta_{\mathbf{x}}^{a, (\mathbf{k}, \ell)}(a, b) = 1 \text{ if and only if } (a, b) \in R_{\mathbf{x}}^a(\mathbf{k}, \ell).$$

The configurations  $\eta_{\mathbf{x}}^{a, (\mathbf{k}, \ell)}$ ,  $(\mathbf{k}, \ell) \in I_a \setminus I_a^*$ , belong to  $\Omega^1$  or form a  $(n-1) \times (n+1)$  rectangle of particles with one extra particle attached to a side of length  $n+1$ . Let  $\Omega^2 = \Omega_{L, K}^2$ , be the set of configurations associated to the pairs  $(\mathbf{k}, \ell)$  in  $I_a^*$ :

$$\Omega^2 = \{\eta_{\mathbf{x}}^{a, (\mathbf{k}, \ell)} : \mathbf{x} \in \Lambda_L, a \in \{\mathfrak{s}, \mathfrak{l}\}, (\mathbf{k}, \ell) \in I_a^*\}.$$

To describe the valleys which can be attained from  $\eta_{\mathbf{x}}^{a, (\mathbf{k}, \ell)}$  we have to define a map from  $\Omega^2$  to  $\Gamma$  which translates by one unit all particles in an external row or column of a rectangle  $R_{\mathbf{x}}^a(\mathbf{k}, \ell)$ . This must be done carefully because the translation of one row may produce a configuration which does not belong to  $\Gamma$ , or a configuration  $\eta_{\mathbf{x}}^{a, (\mathbf{k}', \ell')}$ , where the vector  $(\mathbf{k}', \ell')$  differs from  $(\mathbf{k}, \ell)$  in more than one coordinate.

Denote by  $I_{a,i}^-$  (resp.  $I_{a,i}^+$ ),  $0 \leq i \leq 3$ , the pairs  $(\mathbf{k}, \ell)$  in  $I_a^*$  for which the particle sitting at  $k_i$  (resp.  $\ell_i$ ) jumps to  $k_i - 1$  (resp.  $\ell_i + 1$ ) at rate  $e^{-\beta}$ . The abuse of notation is clear. For instance, by site  $k_0$  we mean the site  $(k_0, 0)$ , or, if  $a = \mathfrak{s}$ , by site  $\ell_2$  we mean site  $(n-1-\ell_2, n)$ . The subsets  $I_{a,i}^\pm$  of  $I_a^*$  are given by

$$\begin{aligned} I_{a,i}^- &= \{(\mathbf{k}, \ell) \in I_a^* : k_i \geq 2 \text{ or } k_i = 1, \ell_{i-1} = n_{i-1}^a\}, \\ I_{a,i}^+ &= \{(\mathbf{k}, \ell) \in I_a^* : \ell_i \leq n_i^a - 1 \text{ or } \ell_i = n_i^a, k_{i+1} = 1\}. \end{aligned}$$

For  $(\mathbf{k}, \ell) \in I_{a,i}^-$ , denote by  $\hat{T}_{a,i}^- \eta_{\mathbf{x}}^{a, (\mathbf{k}, \ell)}$  the configuration obtained from  $\eta_{\mathbf{x}}^{a, (\mathbf{k}, \ell)}$  by moving the particle sitting at  $k_i$  to  $k_i - 1$ , with the same abuse of notation alluded to before. Similarly, for  $(\mathbf{k}, \ell) \in I_{a,i}^+$ , denote by  $\hat{T}_{a,i}^+ \eta_{\mathbf{x}}^{a, (\mathbf{k}, \ell)}$  the configuration obtained from  $\eta_{\mathbf{x}}^{a, (\mathbf{k}, \ell)}$  by moving the particle sitting at  $\ell_i$  to  $\ell_i + 1$ .

Define the map  $T_{a,i}^- : I_{a,i}^- \rightarrow I_a$  by

$$T_{a,i}^-(\mathbf{k}, \ell) = \begin{cases} (\mathbf{k} - \mathbf{e}_i, \ell - \mathbf{e}_i) & \text{if } k_{i+1} \geq 1, \\ (\mathbf{k} - \mathbf{e}_i + \mathbf{e}_{i+1}, \ell) & \text{if } k_{i+1} = 0, \end{cases}$$

where  $\{\mathbf{e}_1, \dots, \mathbf{e}_4\}$  stands for the canonical basis of  $\mathbb{R}^4$ . The map  $T_{a,i}^+ : I_{a,i}^+ \rightarrow I_a$  is defined in an analogous way. Hence, the map  $T_{s,2}^+$  translate to the *left* all particles

on the top row of the rectangle  $R^s$  and the map  $T_{l,3}^-$  translate in the *upward* direction all particles on the leftmost column of  $R^l$ .

The vector  $T_{a,i}^\pm(\mathbf{k}, \ell)$  may not belong to  $I_a^*$  when there are only two particles on one side of a rectangle  $R^a$  and one of them is translated along another side. For example, suppose that  $k_0 = 1, \ell_0 = n - 2, k_1 = 0, \ell_1 = 1$  for a vector  $(\mathbf{k}, \ell) \in I_s^*$ . In this case, necessarily  $k_2 = 1, \ell_2 = n - 2, k_3 = 0, \ell_3 = n - 1$ , and  $T_{s,0}^-(\mathbf{k}, \ell) \notin I_s^*$ . In fact, the configuration  $\eta_{\mathbf{x}}^{s, T_{s,0}^-(\mathbf{k}, \ell)}$  belongs to the set  $\Omega^3$  to be introduced in the next subsection. Similarly, if  $k_1 = 2, \ell_1 = n - 1, k_2 = 0, \ell_2 = 1$  for a vector  $(\mathbf{k}, \ell) \in I_s^*$ ,  $T_{s,1}^-(\mathbf{k}, \ell) \notin I_s^*$ , and  $\eta_{\mathbf{x}}^{s, T_{s,1}^-(\mathbf{k}, \ell)} \in \Omega^1$ .

Fix a vector  $(\mathbf{k}, \ell) \in I_a^*$  such that  $M_i(\mathbf{k}, \ell) = 2$  for some  $0 \leq i \leq 3$ . Denote by  $J_{a,i}(\mathbf{k}, \ell)$  the interval over which the particles on side  $i$  may move:

$$J_{a,i} = J_{a,i}(\mathbf{k}, \ell) = \left\{ 1 - \mathbf{1}\{\ell_{i-1} = n_{i-1}^a\}, \dots, n_i^a + \mathbf{1}\{k_{i+1} \leq 1\} \right\},$$

and by  $T_{a,i}^b(\mathbf{k}, \ell)$ ,  $b, b+1 \in J_{a,i}$ , the vector obtained from  $(\mathbf{k}, \ell)$  by replacing the occupied sites  $k_i, k_i+1$  by the sites  $b, b+1$ . Note that  $T_{a,i}^b(\mathbf{k}, \ell)$  belongs to  $I_a^*$  because we assumed  $n > 3$ . Note also that we did not excluded the possibility that  $b = k_i$  in which case  $T_{a,i}^b(\mathbf{k}, \ell) = (\mathbf{k}, \ell)$ .

The proof of the next result is straightforward and left to the reader. One just needs to identify all possible rate one jumps.

**Lemma 4.3.** *Fix  $0 \leq i \leq 3$ ,  $a \in \{\mathfrak{s}, \mathfrak{l}\}$ ,  $(\mathbf{k}, \ell) \in I_{a,i}^\pm$ , and let  $\xi = \hat{T}_{a,i}^\pm \eta_{\mathbf{x}}^{a, (\mathbf{k}, \ell)}$ . Then, there exists a probability measure  $A_2(\xi, \cdot)$  defined on  $\Gamma$  such that*

$$\lim_{\beta \rightarrow \infty} \mathbf{P}_\xi^\beta [\eta(H(\Gamma)) \in \Pi] = A_2(\xi, \Pi), \quad \Pi \subset \Gamma.$$

Moreover,

$$\begin{aligned} A_2(\hat{T}_{a,i}^\pm \eta_{\mathbf{x}}^{a, (\mathbf{k}, \ell)}, \eta_{\mathbf{x}}^{a, T_{a,i}^\pm(\mathbf{k}, \ell)}) &= \frac{1}{M_i(\mathbf{k}, \ell)}, \\ A_2(\hat{T}_{a,i}^\pm \eta_{\mathbf{x}}^{a, (\mathbf{k}, \ell)}, \eta_{\mathbf{x}}^{a, (\mathbf{k}, \ell)}) &= \frac{M_i(\mathbf{k}, \ell) - 1}{M_i(\mathbf{k}, \ell)} \end{aligned}$$

if  $M_i(\mathbf{k}, \ell) \geq 3$ ; and

$$\begin{aligned} A_2(\hat{T}_{a,i}^- \eta_{\mathbf{x}}^{a, (\mathbf{k}, \ell)}, \eta_{\mathbf{x}}^{a, T_{a,i}^b(\mathbf{k}, \ell)}) &= p(J_{a,i}(\mathbf{k}, \ell), k_i - 1, b), \quad b, b+1 \in J_{a,i}(\mathbf{k}, \ell), \\ A_2(\hat{T}_{a,i}^+ \eta_{\mathbf{x}}^{a, (\mathbf{k}, \ell)}, \eta_{\mathbf{x}}^{a, T_{a,i}^b(\mathbf{k}, \ell)}) &= p(J_{a,i}(\mathbf{k}, \ell), k_i, b), \quad b, b+1 \in J_{a,i}(\mathbf{k}, \ell), \end{aligned}$$

if  $M_i(\mathbf{k}, \ell) = 2$ , where the probability  $p(J, a, c)$  has been introduced just after (4.2).

Denote by  $F(\eta_{\mathbf{x}}^{a, (\mathbf{k}, \ell)})$ ,  $a \in \{\mathfrak{s}, \mathfrak{l}\}$ ,  $(\mathbf{k}, \ell) \in I_a^*$ ,  $\mathbf{x} \in \Lambda_L$ , the set of all configurations  $\hat{T}_{a,i}^\pm \eta_{\mathbf{x}}^{a, (\mathbf{k}, \ell)}$ ,  $0 \leq i \leq 3$ , which appear in the previous lemma. The cardinality of this set depends on  $(\mathbf{k}, \ell)$  and is at most 8. Let

$$Z(\eta_{\mathbf{x}}^{a, (\mathbf{k}, \ell)}) = \sum_{\xi \in F(\eta_{\mathbf{x}}^{a, (\mathbf{k}, \ell)})} \sum_{\zeta \neq \eta_{\mathbf{x}}^{a, (\mathbf{k}, \ell)}} A_2(\xi, \zeta) = \sum_{\xi \in F(\eta_{\mathbf{x}}^{a, (\mathbf{k}, \ell)})} \{1 - A_2(\xi, \eta_{\mathbf{x}}^{a, (\mathbf{k}, \ell)})\}.$$

Note that  $Z(\eta_{\mathbf{x}}^{\mathbf{a},(\mathbf{k},\ell)})$  does not depend on  $\mathbf{x}$  and that

$$\begin{aligned} Z(\eta_{\mathbf{w}}^{\mathbf{a},(\mathbf{k},\ell)}) &= \sum_{i=0}^3 \frac{\mathbf{1}\{M_i(\mathbf{k},\ell) > 2\}}{M_i(\mathbf{k},\ell)} \left\{ \mathbf{1}\{(\mathbf{k},\ell) \in I_{\mathbf{a},i}^-\} + \mathbf{1}\{(\mathbf{k},\ell) \in I_{\mathbf{a},i}^+\} \right\} \\ &+ \sum_{i=0}^3 \mathbf{1}\{M_i(\mathbf{k},\ell) = 2\} \mathbf{1}\{(\mathbf{k},\ell) \in I_{\mathbf{a},i}^-\} [1 - p(J_{\mathbf{a},i}, k_i - 1, k_i)] \\ &+ \sum_{i=0}^3 \mathbf{1}\{M_i(\mathbf{k},\ell) = 2\} \mathbf{1}\{(\mathbf{k},\ell) \in I_{\mathbf{a},i}^+\} [1 - p(J_{\mathbf{a},i}, k_i, k_i)] . \end{aligned}$$

**Proposition 4.4.** Fix  $\mathbf{x} \in \Lambda_L$ ,  $\mathbf{a} \in \{\mathbf{s}, \mathbf{l}\}$ ,  $(\mathbf{k}, \ell) \in I_{\mathbf{a}}^*$ . Then,

- (1) The triple  $(\{\eta_{\mathbf{x}}^{\mathbf{a},(\mathbf{k},\ell)}\}, \{\eta_{\mathbf{x}}^{\mathbf{a},(\mathbf{k},\ell)}\} \cup \Delta_1, \eta_{\mathbf{x}}^{\mathbf{a},(\mathbf{k},\ell)})$  is a valley of depth given by  $\mu_{\beta}(\eta_{\mathbf{x}}^{\mathbf{a},(\mathbf{k},\ell)}) / \text{cap}_{\beta}(\{\eta_{\mathbf{x}}^{\mathbf{a},(\mathbf{k},\ell)}\}, [\{\eta_{\mathbf{x}}^{\mathbf{a},(\mathbf{k},\ell)}\} \cup \Delta_1]^c)$ ;
- (2) Under  $\mathbf{P}_{\eta_{\mathbf{x}}^{\mathbf{a},(\mathbf{k},\ell)}}^{\beta}$ ,  $H(\Gamma \setminus \{\eta_{\mathbf{x}}^{\mathbf{a},(\mathbf{k},\ell)}\})/e^{\beta}$  converges in distribution to an exponential random variable of parameter  $Z(\eta_{\mathbf{x}}^{\mathbf{a},(\mathbf{k},\ell)})$ ;
- (3) For any  $\Pi \subset \Gamma \setminus \{\eta_{\mathbf{x}}^{\mathbf{a},(\mathbf{k},\ell)}\}$ ,

$$\begin{aligned} \lim_{\beta \rightarrow \infty} \mathbf{P}_{\eta_{\mathbf{x}}^{\mathbf{a},(\mathbf{k},\ell)}}^{\beta} [\eta(H(\Gamma \setminus \{\eta_{\mathbf{x}}^{\mathbf{a},(\mathbf{k},\ell)}\})) \in \Pi] \\ = \frac{1}{Z(\eta_{\mathbf{x}}^{\mathbf{a},(\mathbf{k},\ell)})} \sum_{\xi \in F(\eta_{\mathbf{x}}^{\mathbf{a},(\mathbf{k},\ell)})} A_2(\xi, \Pi) =: Q(\eta_{\mathbf{x}}^{\mathbf{a},(\mathbf{k},\ell)}, \Pi) . \end{aligned}$$

*Proof.* Recall [2, Theorem 2.6]. Assumption (2.14) is fulfilled by default and assumption (2.15) follows from the definition of the set  $\Delta_1$ . This proves the first assertion of the proposition.

The proof of the second claim is simpler than the one of the second assertion of Proposition 4.2 if we take  $\tau_1$  as the time of the first jump. With this definition,  $\tau_1$  and  $\eta_{\tau_1}^{\beta}$  are independent random variables by the Markov property,  $\tau_1/e^{\beta}$  converges to an exponential random variable of parameter  $|F_0|$ , where  $F_0 = F_0(\eta_{\mathbf{x}}^{\mathbf{a},(\mathbf{k},\ell)})$  is the set of all configurations which can be attained from  $\eta_{\mathbf{x}}^{\mathbf{a},(\mathbf{k},\ell)}$  by a jump of rate  $e^{-\beta}$ , and  $\eta_{\tau_1}^{\beta}$  converges to a random variable which is uniformly distributed over  $F_0$ .

We did not exhausted in the statement of Lemma 4.3 all configurations of  $F_0$ . For example, if  $\mathbf{a} = \mathbf{s}$ ,  $\mathbf{x} = \mathbf{w}$ ,  $\ell_0 = n - 3$  and  $k_1 \geq 2$ , the particle at  $(n - 2, 1)$  jumps at rate  $e^{-\beta}$  to  $(n - 2, 0)$ . However, under this configuration, the probability of the event  $H(\Gamma) \neq H(\eta_{\mathbf{w}}^{\mathbf{s},(\mathbf{k},\ell)})$  converges to 0 since the unique rate one jump from this configuration is the return to  $\eta_{\mathbf{w}}^{\mathbf{s},(\mathbf{k},\ell)}$ .

By the arguments of Proposition 4.2, starting from  $\eta_{\mathbf{x}}^{\mathbf{a},(\mathbf{k},\ell)}$ ,  $H(\Gamma \setminus \{\eta_{\mathbf{x}}^{\mathbf{a},(\mathbf{k},\ell)}\})/e^{\beta}$  converges in distribution to an exponential random variable of parameter

$$\gamma = \lim_{\beta \rightarrow \infty} \sum_{\xi \in F_0} \mathbf{P}_{\xi}^{\beta} [H(\Gamma) \neq H(\eta_{\mathbf{x}}^{\mathbf{a},(\mathbf{k},\ell)})] .$$

To conclude the proof, it remains to recall the statement of Lemma 4.3, the observation that the configurations in  $F_0 \setminus F(\eta_{\mathbf{x}}^{\mathbf{a},(\mathbf{k},\ell)})$  are irrelevant in the previous sum, and the definition of  $Z(\eta_{\mathbf{x}}^{\mathbf{a},(\mathbf{k},\ell)})$ .

The proof of the third assertion of the proposition is identical to the one of the third claim of Proposition 4.2.  $\square$



As in (4.6), the second assertion of this proposition gives an explicit expression for the depth of the valley presented in the first statement. It follows from Lemma 4.3 that if  $M_i(\mathbf{k}, \ell) > 2$  for some  $0 \leq i \leq 3$ ,

$$Q(\eta_{\mathbf{x}}^{\mathbf{a},(\mathbf{k},\ell)}, \eta_{\mathbf{x}}^{\mathbf{a},T_{\mathbf{a},i}^{\pm}(\mathbf{k},\ell)}) = \frac{1}{Z(\eta_{\mathbf{x}}^{\mathbf{a},(\mathbf{k},\ell)})} \frac{\mathbf{1}\{(\mathbf{k}, \ell) \in I_{\mathbf{a},i}^{\pm}\}}{M_i(\mathbf{k}, \ell)}$$

and if  $M_i(\mathbf{k}, \ell) = 2$ ,

$$Q(\eta_{\mathbf{x}}^{\mathbf{a},(\mathbf{k},\ell)}, \eta_{\mathbf{x}}^{\mathbf{a},T_{\mathbf{a},i}^b(\mathbf{k},\ell)}) = \frac{1}{Z(\eta_{\mathbf{x}}^{\mathbf{a},(\mathbf{k},\ell)})} p_{\mathbf{a}}(\mathbf{k}, \ell, i, b), \quad b, b+1 \in J_{\mathbf{a},i}(\mathbf{k}, \ell),$$

where

$$p_{\mathbf{a}}(\mathbf{k}, \ell, i, b) = \mathbf{1}\{(\mathbf{k}, \ell) \in I_{\mathbf{a},i}^-\} p(J_{\mathbf{a},i}, k_i - 1, b) + \mathbf{1}\{(\mathbf{k}, \ell) \in I_{\mathbf{a},i}^+\} p(J_{\mathbf{a},i}, k_i, b).$$

It follows from Proposition 4.4 and Lemma 4.3 that starting from a configuration  $\zeta \in \Omega^2$  the process  $\eta_t^\beta$  reaches  $\Gamma$  only in a configuration of  $\Omega^1 \cup \Omega^2$  or in a configuration in which all sites of a  $(n-1) \times (n+1)$  rectangle are occupied and an extra particle is attached to a side of length  $n+1$ . To pursue our analysis, we have to introduce this new set of valleys.

**4.3. The valleys  $\mathcal{E}_{\mathbf{x}}^{\mathbf{a},i}$ .** The arguments of this subsection are similar to the ones of Subsection 4.1. Let  $T^{\mathbf{l}}, T^{\mathbf{s}}$  be the rectangles  $T^{\mathbf{l}} = \{0, \dots, n\} \times \{0, \dots, n-2\}$ ,  $T^{\mathbf{s}} = \{0, \dots, n-2\} \times \{0, \dots, n\}$ . Denote by  $T_{\mathbf{x}}^{\mathbf{a}}$ ,  $\mathbf{a} \in \{\mathbf{s}, \mathbf{l}\}$ ,  $\mathbf{x} \in \Lambda_L$ , the rectangle  $T^{\mathbf{a}}$  translated by  $\mathbf{x}$ :  $T_{\mathbf{x}}^{\mathbf{a}} = \mathbf{x} + T^{\mathbf{a}}$ , and by  $\eta_{\mathbf{x},\mathbf{a}}$  the configuration in which all sites of  $T_{\mathbf{x}}^{\mathbf{a}}$  are occupied. Note that  $\eta_{\mathbf{x},\mathbf{a}}$  belongs to  $\Omega_{L,K-1}$  and not to  $\Omega_{L,K}$ .

For  $\mathbf{a} \in \{\mathbf{s}, \mathbf{l}\}$ ,  $\mathbf{x} \in \Lambda_L$ ,  $\mathbf{z} \in \partial T_{\mathbf{x}}^{\mathbf{a}}$ , denote by  $\eta_{\mathbf{x},\mathbf{a}}^{\mathbf{z}}$  the configuration in which all sites of the rectangle  $T_{\mathbf{x}}^{\mathbf{a}}$  and the site  $\mathbf{z}$  are occupied:  $\eta_{\mathbf{x},\mathbf{a}}^{\mathbf{z}} = \eta_{\mathbf{x},\mathbf{a}} + \mathfrak{d}_{\mathbf{z}}$ , where  $\mathfrak{d}_y$ ,  $y \in \Lambda_L$ , is the configuration with a unique particle at  $y$  and summation of configurations is performed componentwise. Let  $\Omega^3 = \Omega_{L,K}^3$  be the set of such configurations:

$$\Omega^3 = \{\eta_{\mathbf{x},\mathbf{a}}^{\mathbf{z}} : \mathbf{a} \in \{\mathbf{s}, \mathbf{l}\}, \mathbf{x} \in \Lambda_L, \mathbf{z} \in \partial T_{\mathbf{x}}^{\mathbf{a}}\}.$$

Denote by  $\partial_j T_{\mathbf{x}}^{\mathbf{a}}$ ,  $0 \leq j \leq 3$ , the  $j$ -th boundary of  $T_{\mathbf{x}}^{\mathbf{a}}$ :

$$\begin{aligned} \partial_j T_{\mathbf{x}}^{\mathbf{a}} &= \{\mathbf{z} \in \partial T_{\mathbf{x}}^{\mathbf{a}} : \exists \mathbf{y} \in T_{\mathbf{x}}^{\mathbf{a}}; \mathbf{y} - \mathbf{z} = (1-j)e_2\} \quad j = 0, 2, \\ \partial_j T_{\mathbf{x}}^{\mathbf{a}} &= \{\mathbf{z} \in \partial T_{\mathbf{x}}^{\mathbf{a}} : \exists \mathbf{y} \in T_{\mathbf{x}}^{\mathbf{a}}; \mathbf{y} - \mathbf{z} = (j-2)e_1\} \quad j = 1, 3. \end{aligned}$$

Let  $\mathcal{E}_{\mathbf{x}}^{\mathbf{a},j}$  be the set of configurations in which all sites of the set  $T_{\mathbf{x}}^{\mathbf{a}}$  are occupied with an extra particle at some location of  $\partial_j T_{\mathbf{x}}^{\mathbf{a}}$ :

$$\mathcal{E}_{\mathbf{x}}^{\mathbf{a},j} = \{\eta_{\mathbf{x},\mathbf{a}}^{\mathbf{z}} \in \Omega^3 : \mathbf{z} \in \partial_j T_{\mathbf{x}}^{\mathbf{a}}\}.$$

The process  $\{\eta_t^\beta : t \geq 0\}$  can reach any configuration  $\xi \in \mathcal{E}_{\mathbf{x}}^{\mathbf{a},j}$  from any configuration  $\eta \in \mathcal{E}_{\mathbf{x}}^{\mathbf{a},j}$  with rate one jumps. Hence, in the terminology introduced in [4], the sets  $\mathcal{E}_{\mathbf{x}}^{\mathbf{a},j}$  are equivalent classes. The main result of this subsection states that for any configuration  $\xi \in \mathcal{E}_{\mathbf{x}}^{\mathbf{a},j}$ , the triples  $(\mathcal{E}_{\mathbf{x}}^{\mathbf{a},j}, \mathcal{E}_{\mathbf{x}}^{\mathbf{a},j} \cup \Delta_1, \xi)$  are valleys.

By symmetry, the distribution of  $\eta(H(\Gamma \setminus \mathcal{E}_{\mathbf{x}}^{\mathbf{a},j}))$  can be obtained from the one of  $\eta(H(\Gamma \setminus \mathcal{E}_{\mathbf{w}}^{\mathbf{s},0}))$  or from the one of  $\eta(H(\Gamma \setminus \mathcal{E}_{\mathbf{w}}^{\mathbf{s},1}))$ . Denote by  $F_{\mathbf{x}}^{\mathbf{a},j}$ , the configurations which do not belong to  $\mathcal{E}_{\mathbf{x}}^{\mathbf{a},j}$ , but which can be reached from a configuration in  $\mathcal{E}_{\mathbf{x}}^{\mathbf{a},j}$  by performing a jump of rate  $e^{-\beta}$ . The set  $F_{\mathbf{w}}^{\mathbf{s},0}$  has the following  $n+3$  elements. There are  $n+1$  configurations obtained when the bottom particle detaches itself from the others:  $\eta_{\mathbf{w},\mathbf{s}} + \mathfrak{d}_{\mathbf{z}}$ , where  $\mathbf{z} \in J_2 = \{(-1, -1), (a, -2), (n-1, -1) : 0 \leq a \leq n-2\}$ . There is a configuration in  $F_{\mathbf{w}}^{\mathbf{s},0}$  which is obtained when the bottom

particle is at  $(1, -1)$  and the particle at  $\mathbf{w}$  moves to  $\mathbf{w} - e_2$ :  $\sigma^{\mathbf{w}, \mathbf{w} - e_2} \eta_{\mathbf{w}, \mathbf{s}}^{(1, -1)}$ . The last configuration of  $F_{\mathbf{w}}^{s, 0}$  is obtained when the bottom particle is at  $(n-3, -1)$  and the particle at  $\mathbf{w}_1 - e_1$  moves to  $\mathbf{w}_1 - e_1 - e_2$ :  $\sigma^{\mathbf{w}_1 - e_1, \mathbf{w}_1 - e_1 - e_2} \eta_{\mathbf{w}, \mathbf{s}}^{(n-3, -1)}$ .

**Lemma 4.5.** *Fix  $\mathbf{x} \in \Lambda_L$ ,  $\mathbf{a} \in \{\mathbf{s}, \mathbf{l}\}$  and  $0 \leq j \leq 3$ . For each  $\xi \in F_{\mathbf{x}}^{\mathbf{a}, j}$ , there exists a probability measure  $A_{\mathbf{x}}^{\mathbf{a}, j}(\xi, \cdot)$  defined on  $\Gamma$  such that*

$$\lim_{\beta \rightarrow \infty} \mathbf{P}_{\xi}^{\beta} [\eta(H_{\Gamma}) \in \Pi] = A_{\mathbf{x}}^{\mathbf{a}, j}(\xi, \Pi), \quad \Pi \subset \Gamma.$$

Moreover, if  $\xi \in F_{\mathbf{w}}^{s, 0}$ ,

$$A_{\mathbf{w}}^{s, 0}(\eta_{\mathbf{w}, \mathbf{s}} + \mathbf{d}_z, \partial_j T_{\mathbf{w}}^s) = p(\mathbf{z}, \partial_j T_{\mathbf{w}}^s, \partial T_{\mathbf{w}}^s), \quad 0 \leq j \leq 3, \quad \mathbf{z} \in J_2.$$

$$A_{\mathbf{w}}^{s, 0}(\sigma^{\mathbf{w}, \mathbf{w} - e_2} \eta_{\mathbf{w}, \mathbf{s}}^{(1, -1)}, \Pi) = \begin{cases} \frac{1}{n+1} & \text{if } \Pi = \{\sigma^{\mathbf{w}_3 + e_2, \mathbf{w} - e_2} \eta_{\mathbf{w}, \mathbf{s}}^{(1, -1)}\}, \\ \frac{n}{n+1} & \text{if } \Pi = \{\eta_{\mathbf{w}, \mathbf{s}}^{(1, -1)}\}. \end{cases}$$

$$A_{\mathbf{w}}^{s, 0}(\xi, \Pi) = \begin{cases} \frac{1}{n+1} & \text{if } \Pi = \{\sigma^{\mathbf{w}_2 - e_1 + e_2, \mathbf{w}_1 - e_1 - e_2} \eta_{\mathbf{w}, \mathbf{s}}^{(n-3, -1)}\}, \\ \frac{n}{n+1} & \text{if } \Pi = \{\eta_{\mathbf{w}, \mathbf{s}}^{(n-3, -1)}\}, \end{cases}$$

if  $\xi = \sigma^{\mathbf{w}_1 - e_1, \mathbf{w}_1 - e_1 - e_2} \eta_{\mathbf{w}, \mathbf{s}}^{(n-3, -1)}$ .

The proof of the previous lemma is simpler than the one of Lemma 4.1 and left to the reader. Define

$$Z(\mathcal{E}_{\mathbf{x}}^{\mathbf{a}, j}) = \sum_{\xi \in F_{\mathbf{x}}^{\mathbf{a}, j}} A_{\mathbf{x}}^{\mathbf{a}, j}(\xi, (\mathcal{E}_{\mathbf{x}}^{\mathbf{a}, j})^c) = \sum_{\xi \in F_{\mathbf{x}}^{\mathbf{a}, j}} \{1 - A_{\mathbf{x}}^{\mathbf{a}, j}(\xi, \mathcal{E}_{\mathbf{x}}^{\mathbf{a}, j})\}.$$

By the previous lemma,

$$Z(\mathcal{E}_{\mathbf{w}}^{s, 0}) = \frac{2}{n+1} + \sum_{\mathbf{y} \in J_2} [1 - p(\mathbf{y}, \partial_0 T_{\mathbf{w}}^s, \partial T_{\mathbf{w}}^s)].$$

**Proposition 4.6.** *Fix  $0 \leq j \leq 3$ ,  $\mathbf{a} \in \{\mathbf{l}, \mathbf{s}\}$ ,  $\mathbf{x} \in \Lambda_L$ .*

- (1) *For every  $\xi \in \mathcal{E}_{\mathbf{x}}^{\mathbf{a}, j}$ , the triple  $(\mathcal{E}_{\mathbf{x}}^{\mathbf{a}, j}, \mathcal{E}_{\mathbf{x}}^{\mathbf{a}, j} \cup \Delta_1, \xi)$  is a valley of depth  $\mu_{\beta}(\mathcal{E}_{\mathbf{x}}^{\mathbf{a}, j}) / \text{cap}_{\beta}(\mathcal{E}_{\mathbf{x}}^{\mathbf{a}, j}, [\mathcal{E}_{\mathbf{x}}^{\mathbf{a}, j} \cup \Delta_1]^c)$ ;*
- (2) *For any  $\xi \in \mathcal{E}_{\mathbf{x}}^{\mathbf{a}, j}$ , under  $\mathbf{P}_{\xi}^{\beta}$ ,  $H(\Gamma \setminus \mathcal{E}_{\mathbf{x}}^{\mathbf{a}, j}) / e^{\beta}$  converges in distribution to an exponential random variable of parameter  $Z(\mathcal{E}_{\mathbf{x}}^{\mathbf{a}, j})$ ;*
- (3) *For any  $\xi \in \mathcal{E}_{\mathbf{x}}^{\mathbf{a}, j}$ ,  $\Pi \subset \Gamma \setminus \mathcal{E}_{\mathbf{x}}^{\mathbf{a}, j}$ ,*

$$\lim_{\beta \rightarrow \infty} \mathbf{P}_{\xi}^{\beta} [\eta(H(\Gamma \setminus \mathcal{E}_{\mathbf{x}}^{\mathbf{a}, j})) \in \Pi] = \frac{1}{Z(\mathcal{E}_{\mathbf{x}}^{\mathbf{a}, j})} \sum_{\eta \in F_{\mathbf{x}}^{\mathbf{a}, j}} A_{\mathbf{x}}^{\mathbf{a}, j}(\eta, \Pi) =: Q(\mathcal{E}_{\mathbf{x}}^{\mathbf{a}, j}, \Pi).$$

The proof of this proposition is similar to the one of Proposition 4.2, with  $\tau_1$  defined as the first time the process leaves the set  $\mathcal{E}_{\mathbf{x}}^{\mathbf{a}, j}$ . Remark (4.6) concerning the explicit formula for the depth of the valley appearing in the first statement of Proposition 4.6 also holds.

It follows from the previous two results that starting from a configuration  $\zeta \in \Omega^3$  the process  $\eta_t^{\beta}$  reaches  $\Gamma$  only in a configuration of  $\Omega^2 \cup \Omega^3$  or in a configuration in which all sites of a  $(n-3) \times n$  rectangle are occupied with  $3n$  extra particles attached to the boundary. This is the last set of valleys which needs to be examined.

4.4. **The valleys**  $\{\zeta_{\mathbf{x}}^{\mathbf{a},(\mathbf{k},\ell)}\}$ . The arguments of this subsection are similar to the ones of Subsection 4.2. Let  $R^{2,\mathbf{l}}, R^{2,\mathbf{s}}$  be the rectangles  $R^{2,\mathbf{l}} = \{1, \dots, n\} \times \{1, \dots, n-3\}$ ,  $R^{2,\mathbf{s}} = \{1, \dots, n-3\} \times \{1, \dots, n\}$ . Let  $n_0^{2,\mathbf{s}} = n_2^{2,\mathbf{s}} = n-3$ ,  $n_1^{2,\mathbf{s}} = n_3^{2,\mathbf{s}} = n$  be the length of the sides of the standing rectangle  $R^{2,\mathbf{s}}$ . Similarly, denote by  $n_i^{2,\mathbf{l}}$ ,  $0 \leq i \leq 3$ , the length of the sides of the lying rectangle  $R^{2,\mathbf{l}}$ :  $n_i^{2,\mathbf{l}} = n_{i+1}^{2,\mathbf{s}}$ , where the sum over the index  $i$  is performed modulo 4.

Denote by  $\mathbb{I}_{2,\mathbf{a}}$ ,  $\mathbf{a} \in \{\mathbf{s}, \mathbf{l}\}$ , the set of pairs  $(\mathbf{k}, \ell)$  such that

- $0 \leq k_i \leq \ell_i \leq n_i^{2,\mathbf{a}}$ ,
- If  $k_j = 0$ , then  $\ell_{j-1} = n_{j-1}^{2,\mathbf{a}}$ .

For  $(\mathbf{k}, \ell) \in \mathbb{I}_{2,\mathbf{a}}$ ,  $\mathbf{a} \in \{\mathbf{s}, \mathbf{l}\}$ , let  $R^{2,\mathbf{l}}(\mathbf{k}, \ell)$ ,  $R^{2,\mathbf{s}}(\mathbf{k}, \ell)$  be the sets

$$\begin{aligned} R^{2,\mathbf{l}}(\mathbf{k}, \ell) &= R^{2,\mathbf{l}} \cup \{(a, 0) : k_0 \leq a \leq \ell_0\} \cup \{(n+1, b) : k_1 \leq b \leq \ell_1\} \cup \\ &\quad \cup \{(n+1-a, n-2) : k_2 \leq a \leq \ell_2\} \cup \{(0, n-2-b) : k_3 \leq b \leq \ell_3\}, \\ R^{2,\mathbf{s}}(\mathbf{k}, \ell) &= R^{2,\mathbf{s}} \cup \{(a, 0) : k_0 \leq a \leq \ell_0\} \cup \{(n-2, b) : k_1 \leq b \leq \ell_1\} \cup \\ &\quad \cup \{(n-2-a, n+1) : k_2 \leq a \leq \ell_2\} \cup \{(0, n+1-b) : k_3 \leq b \leq \ell_3\}. \end{aligned}$$

Denote by  $I_{2,\mathbf{a}}$ ,  $\mathbf{a} \in \{\mathbf{s}, \mathbf{l}\}$ , the set of pairs  $(\mathbf{k}, \ell) \in \mathbb{I}_{2,\mathbf{a}}$  such that  $|R^{2,\mathbf{a}}(\mathbf{k}, \ell)| = n^2$ . For  $(\mathbf{k}, \ell) \in I_{2,\mathbf{a}}$ , denote by  $M_i^{2,\mathbf{a}}(\mathbf{k}, \ell)$  the number of particles attached to the side  $i$  of the rectangle  $R^{2,\mathbf{a}}(\mathbf{k}, \ell)$ :

$$M_i^{2,\mathbf{a}}(\mathbf{k}, \ell) = \begin{cases} \ell_i - k_i + 1 & \text{if } k_{i+1} \geq 1, \\ \ell_i - k_i + 2 & \text{if } k_{i+1} = 0. \end{cases}$$

Clearly, for  $(\mathbf{k}, \ell) \in I_{2,\mathbf{a}}$ ,  $\sum_{0 \leq i \leq 3} M_i^{2,\mathbf{a}}(\mathbf{k}, \ell) = 3n + A$ , where  $A$  is the number of occupied corners, which are counted twice since they are attached to two sides.

Denote by  $I_{2,\mathbf{a}}^* \subset I_{2,\mathbf{a}}$ , the set of pairs  $(\mathbf{k}, \ell) \in I_{2,\mathbf{a}}$  whose rectangles  $R^{2,\mathbf{a}}(\mathbf{k}, \ell)$  have at least two particles on each side:  $M_i^{2,\mathbf{a}}(\mathbf{k}, \ell) \geq 2$ ,  $0 \leq i \leq 3$ . Note that if  $(\mathbf{k}, \ell)$  belongs to  $I_{2,\mathbf{a}}^*$ , for all  $\mathbf{x} \in R^{2,\mathbf{a}}(\mathbf{k}, \ell)$ , there exist  $\mathbf{y}, \mathbf{z} \in R^{2,\mathbf{a}}(\mathbf{k}, \ell)$ ,  $\mathbf{y} \neq \mathbf{z}$ , with the property  $|\mathbf{x} - \mathbf{y}| = |\mathbf{x} - \mathbf{z}| = 1$ .

For  $(\mathbf{k}, \ell) \in I_{2,\mathbf{a}}$ ,  $\mathbf{a} \in \{\mathbf{s}, \mathbf{l}\}$ ,  $\mathbf{x} \in \Lambda_L$ , let  $R_{\mathbf{x}}^{2,\mathbf{a}}(\mathbf{k}, \ell) = \mathbf{x} + R^{2,\mathbf{a}}(\mathbf{k}, \ell)$ , and let  $\zeta_{\mathbf{x}}^{\mathbf{a},(\mathbf{k},\ell)}$  represent the configurations defined by

$$\zeta_{\mathbf{x}}^{\mathbf{a},(\mathbf{k},\ell)}(a, b) = 1 \text{ if and only if } (a, b) \in R_{\mathbf{x}}^{2,\mathbf{a}}(\mathbf{k}, \ell).$$

The configurations  $\zeta_{\mathbf{x}}^{\mathbf{a},(\mathbf{k},\ell)}$ ,  $(\mathbf{k}, \ell) \in I_{2,\mathbf{a}}^*$ , have at least four particles attached to the longer side, and the configurations  $\zeta_{\mathbf{x}}^{\mathbf{a},(\mathbf{k},\ell)}$ ,  $(\mathbf{k}, \ell) \in I_{2,\mathbf{a}} \setminus I_{2,\mathbf{a}}^*$ , belong to  $\Omega^3$ , forming a  $(n-1) \times (n+1)$  rectangle of particles with one extra particle attached to a side of length  $n-1$ . Let  $\Omega^4 = \Omega_{L,K}^4$ , be the set of configurations associated to the pairs  $(\mathbf{k}, \ell)$  in  $I_{2,\mathbf{a}}^*$ :

$$\Omega^4 = \{\zeta_{\mathbf{x}}^{\mathbf{a},(\mathbf{k},\ell)} : \mathbf{x} \in \Lambda_L, \mathbf{a} \in \{\mathbf{s}, \mathbf{l}\}, (\mathbf{k}, \ell) \in I_{2,\mathbf{a}}^*\}.$$

Denote by  $I_{2,\mathbf{a},i}^\pm$ ,  $\mathbf{a} \in \{\mathbf{s}, \mathbf{l}\}$ ,  $0 \leq i \leq 3$ , the subset of  $I_{2,\mathbf{a}}^*$  defined by

$$\begin{aligned} I_{2,\mathbf{a},i}^- &= \{(\mathbf{k}, \ell) \in I_{2,\mathbf{a}}^* : k_i \geq 2 \text{ or } k_i = 1, \ell_{i-1} = n_{i-1}^{2,\mathbf{a}}\}, \\ I_{2,\mathbf{a},i}^+ &= \{(\mathbf{k}, \ell) \in I_{2,\mathbf{a}}^* : \ell_i \leq n_i^{2,\mathbf{a}} - 1 \text{ or } \ell_i = n_i^{2,\mathbf{a}}, k_{i+1} = 1\}. \end{aligned}$$

For  $(\mathbf{k}, \ell) \in I_{2,a,i}^-$ , denote by  $\hat{T}_{2,a,i}^{\pm} \zeta_{\mathbf{x}}^{a,(\mathbf{k},\ell)}$  the configuration obtained from  $\zeta_{\mathbf{x}}^{a,(\mathbf{k},\ell)}$  by moving the particle sitting at  $k_i$  to  $k_i - 1$ . As in Subsection 4.2, the abuse of notation is clear. Similarly, for  $(\mathbf{k}, \ell) \in I_{2,a,i}^+$ , denote by  $\hat{T}_{2,a,i}^{\pm} \zeta_{\mathbf{x}}^{a,(\mathbf{k},\ell)}$  the configuration obtained from  $\zeta_{\mathbf{x}}^{a,(\mathbf{k},\ell)}$  by moving the particle sitting at  $\ell_i$  to  $\ell_i + 1$ .

Define the map  $T_{2,a,i}^- : I_{2,a,i}^- \rightarrow I_{2,a}$  by

$$T_{2,a,i}^-(\mathbf{k}, \ell) = \begin{cases} (\mathbf{k} - \mathbf{e}_i, \ell - \mathbf{e}_i) & \text{if } k_{i+1} \geq 1, \\ (\mathbf{k} - \mathbf{e}_i + \mathbf{e}_{i+1}, \ell) & \text{if } k_{i+1} = 0. \end{cases}$$

The map  $T_{2,a,i}^+ : I_{2,a,i}^+ \rightarrow I_{2,a}$  is defined in an analogous way.

The vector  $T_{2,a,i}^{\pm}(\mathbf{k}, \ell)$  may not belong to  $I_{2,a}^*$  when there are only two particles on one side of a rectangle  $R^{2,a}$  and one of them is translated along another side. Since there are at least four particles attached to the longer sides of the rectangle, this may happen only in the shorter sides of the rectangles. In this case the configuration associated to the vector  $T_{2,a,i}^{\pm}(\mathbf{k}, \ell)$  belongs to  $\Omega^3$ .

Fix a vector  $(\mathbf{k}, \ell) \in I_{2,a}^*$  such that  $M_i^{2,a}(\mathbf{k}, \ell) = 2$  for some  $0 \leq i \leq 3$ . Denote by  $J_{2,a,i}(\mathbf{k}, \ell)$  the interval over which the particles on side  $i$  may move:

$$J_{2,a,i} = J_{2,a,i}(\mathbf{k}, \ell) = \left\{ 1 - \mathbf{1}\{\ell_{i-1} = n_{i-1}^{2,a}\}, \dots, n_i^{2,a} + \mathbf{1}\{k_{i+1} \leq 1\} \right\},$$

and by  $T_{2,a,i}^b(\mathbf{k}, \ell)$ ,  $b, b+1 \in J_{2,a,i}$ , the vector obtained from  $(\mathbf{k}, \ell)$  by replacing the occupied sites  $(k_i, k_i + 1)$  by  $(b, b+1)$ . Note that  $T_{2,a,i}^b(\mathbf{k}, \ell)$  always belongs to  $I_{2,a}^*$ , and that we did not excluded the possibility that  $b = k_i$  in which case  $T_{2,a,i}^b(\mathbf{k}, \ell) = (\mathbf{k}, \ell)$ .

**Lemma 4.7.** Fix  $\mathbf{a} \in \{\mathfrak{s}, \mathfrak{l}\}$ ,  $0 \leq i \leq 3$ , and  $(\mathbf{k}, \ell) \in I_{2,a,i}^{\pm}$ , and let  $\xi = \hat{T}_{2,a,i}^{\pm} \zeta_{\mathbf{x}}^{a,(\mathbf{k},\ell)}$ . Then, there exists a probability measure  $A_4(\xi, \cdot)$  defined on  $\Gamma$  such that

$$\lim_{\beta \rightarrow \infty} \mathbf{P}_{\xi}^{\beta}[\eta(H(\Gamma)) \in \Pi] = A_4(\xi, \Pi), \quad \Pi \subset \Gamma.$$

Moreover,

$$\begin{aligned} A_4(\hat{T}_{2,a,i}^{\pm} \zeta_{\mathbf{x}}^{a,(\mathbf{k},\ell)}, \zeta_{\mathbf{x}}^{a, T_{2,a,i}^{\pm}(\mathbf{k},\ell)}) &= \frac{1}{M_i^{2,a}(\mathbf{k}, \ell)}, \\ A_4(\hat{T}_{2,a,i}^{\pm} \zeta_{\mathbf{x}}^{a,(\mathbf{k},\ell)}, \zeta_{\mathbf{x}}^{a,(\mathbf{k},\ell)}) &= \frac{M_i^{2,a}(\mathbf{k}, \ell) - 1}{M_i^{2,a}(\mathbf{k}, \ell)} \end{aligned}$$

if  $M_i^{2,a}(\mathbf{k}, \ell) \geq 3$ .

$$\begin{aligned} A_4(\hat{T}_{2,a,i}^- \zeta_{\mathbf{x}}^{a,(\mathbf{k},\ell)}, \zeta_{\mathbf{x}}^{a, T_{2,a,i}^b(\mathbf{k},\ell)}) &= p(J_{2,a,i}(\mathbf{k}, \ell), k_i - 1, b), \\ A_4(\hat{T}_{2,a,i}^+ \zeta_{\mathbf{x}}^{a,(\mathbf{k},\ell)}, \zeta_{\mathbf{x}}^{a, T_{2,a,i}^b(\mathbf{k},\ell)}) &= p(J_{2,a,i}(\mathbf{k}, \ell), k_i, b), \end{aligned}$$

for  $b, b+1 \in J_{2,a,i}(\mathbf{k}, \ell)$  if  $M_i^{2,a}(\mathbf{k}, \ell) = 2$ , where the probability  $p(J, a, c)$  has been introduced just after (4.2).

Denote by  $F_{2,a}(\mathbf{k}, \ell)$ ,  $(\mathbf{k}, \ell) \in I_{2,a}^*$ , the set of all configurations  $\hat{T}_{2,a,i}^{\pm} \zeta_{\mathbf{x}}^{a,(\mathbf{k},\ell)}$ ,  $0 \leq i \leq 3$ . The cardinality of this set depends on  $(\mathbf{k}, \ell)$  and is at most 8. Let

$$Z(\zeta_{\mathbf{x}}^{a,(\mathbf{k},\ell)}) = \sum_{\xi \in F_{2,a}(\mathbf{k}, \ell)} A_4(\xi, \{\zeta_{\mathbf{x}}^{a,(\mathbf{k},\ell)}\}^c) = \sum_{\xi \in F_{2,a}(\mathbf{k}, \ell)} \{1 - A_4(\xi, \zeta_{\mathbf{x}}^{a,(\mathbf{k},\ell)})\}.$$

**Proposition 4.8.** Fix  $\mathbf{x} \in \Lambda_L$ ,  $\mathbf{a} \in \{\mathbf{l}, \mathbf{s}\}$ ,  $(\mathbf{k}, \ell) \in I_{2,\mathbf{a}}^*$ . Then,

- (1) The triple  $(\{\zeta_{\mathbf{x}}^{\mathbf{a},(\mathbf{k},\ell)}\}, \{\zeta_{\mathbf{x}}^{\mathbf{a},(\mathbf{k},\ell)}\} \cup \Delta_1, \zeta_{\mathbf{x}}^{\mathbf{a},(\mathbf{k},\ell)})$  is a valley of depth  $\mu_{\beta}(\zeta_{\mathbf{x}}^{\mathbf{a},(\mathbf{k},\ell)})/\text{cap}_{\beta}(\{\zeta_{\mathbf{x}}^{\mathbf{a},(\mathbf{k},\ell)}\}, [\{\zeta_{\mathbf{x}}^{\mathbf{a},(\mathbf{k},\ell)}\} \cup \Delta_1]^c)$ ;
- (2) Under  $\mathbf{P}_{\zeta_{\mathbf{x}}^{\mathbf{a},(\mathbf{k},\ell)}}^{\beta}$ ,  $H(\Gamma \setminus \{\zeta_{\mathbf{x}}^{\mathbf{a},(\mathbf{k},\ell)}\})/e^{\beta}$  converges in distribution to an exponential random variable of parameter  $Z(\zeta_{\mathbf{x}}^{\mathbf{a},(\mathbf{k},\ell)})$ ;
- (3) For any  $\Pi \subset \Gamma \setminus \{\zeta_{\mathbf{x}}^{\mathbf{a},(\mathbf{k},\ell)}\}$ ,

$$\begin{aligned} & \lim_{\beta \rightarrow \infty} \mathbf{P}_{\zeta_{\mathbf{x}}^{\mathbf{a},(\mathbf{k},\ell)}}^{\beta} [\eta(H(\Gamma \setminus \{\zeta_{\mathbf{x}}^{\mathbf{a},(\mathbf{k},\ell)}\})) \in \Pi] \\ &= \frac{1}{Z(\zeta_{\mathbf{x}}^{\mathbf{a},(\mathbf{k},\ell)})} \sum_{\xi \in F_{2,\mathbf{a}}(\mathbf{k},\ell)} A_4(\xi, \Pi) =: Q(\zeta_{\mathbf{x}}^{\mathbf{a},(\mathbf{k},\ell)}, \Pi). \end{aligned}$$

## 5. TUNNELING BEHAVIOR AMONG SHALLOW VALLEYS

We examine in this section the evolution of the Markov process  $\{\eta_t^{\beta} : t \geq 0\}$  in the time scale  $e^{\beta}$  among the shallow valleys introduced in the previous section. We first introduce a family of deep valleys or traps.

**Lemma 5.1.** Fix  $\mathbf{x} \in \Lambda_L$ . The triple  $(\{\eta^{\mathbf{x}}\}, \{\eta^{\mathbf{x}}\} \cup \Delta_1, \eta^{\mathbf{x}})$  is a valley of depth  $\mu_{\beta}(\eta^{\mathbf{x}})/\text{cap}_{\beta}(\{\eta^{\mathbf{x}}\}, [\{\eta^{\mathbf{x}}\} \cup \Delta_1]^c)$ .

This result follows from [2, Theorem 2.6]. Up to this point, we introduced five types of disjoint subsets of  $\Omega_{L,K}$ :

- $\{\eta^{\mathbf{x}}\}, \mathbf{x} \in \Lambda_L$ ;
- $\mathcal{E}_{\mathbf{x}}^{i,j}, 0 \leq i, j \leq 3, \mathbf{x} \in \Lambda_L$ ;
- $\{\eta_{\mathbf{x}}^{\mathbf{a},(\mathbf{k},\ell)}\}, \mathbf{x} \in \Lambda_L, \mathbf{a} \in \{\mathbf{l}, \mathbf{s}\}, (\mathbf{k}, \ell) \in I_{\mathbf{a}}^*$ ;
- $\mathcal{E}_{\mathbf{x}}^{\mathbf{a},i}, \mathbf{a} \in \{\mathbf{l}, \mathbf{s}\}, 0 \leq i \leq 3, \mathbf{x} \in \Lambda_L$ ;
- $\{\zeta_{\mathbf{x}}^{\mathbf{a},(\mathbf{k},\ell)}\}, \mathbf{x} \in \Lambda_L, \mathbf{a} \in \{\mathbf{l}, \mathbf{s}\}, (\mathbf{k}, \ell) \in I_{2,\mathbf{a}}^*$ .

Denote by  $\mathcal{E}_1, \dots, \mathcal{E}_{\kappa}$  an enumeration of these sets. In this enumeration we shall assume that  $\mathcal{E}_1 = \{\eta^{\mathbf{w}}\}$  and that the first  $|\Lambda_L|$  sets correspond to the square configurations: for  $1 \leq i \leq |\Lambda_L|$ ,  $\mathcal{E}_i = \{\eta^{\mathbf{x}_i}\}$  for some  $\mathbf{x}_i \in \Lambda_L$ . Some sets  $\mathcal{E}_j$  are singletons, as the first  $|\Lambda_L|$  sets, and some are not, as the set  $\mathcal{E}_{|\Lambda_L|+1} = \mathcal{E}_{\mathbf{w}}^{0,0}$ . Let  $\mathcal{E} = \cup_{1 \leq j \leq \kappa} \mathcal{E}_j$  be the union of all subsets and let  $\check{\mathcal{E}}_j = \cup_{i \neq j} \mathcal{E}_i$ . For  $1 \leq i \leq |\Lambda_L|$ , we sometimes denote  $\mathcal{E}_i = \{\eta^{\mathbf{x}}\}$  by  $\mathcal{E}_{\mathbf{x}}$ .

We have shown in the first section of this article that  $\Omega^0$  is the set of ground states of the energy  $\mathbb{H}$  in  $\Omega_{L,K}$ . Denote by  $\mathfrak{G}_1$  the set of configurations which minimize the energy over  $\Omega_{L,K} \setminus \Omega^0$ . All sets  $\mathcal{E}_j, j > |\Lambda_L|$ , are contained in  $\mathfrak{G}_1$ , but it is easy to exhibit configurations in  $\mathfrak{G}_1$  which do not belong to  $\mathcal{E}$ .

With this notation,  $\Gamma = \Omega^0 \cup \mathfrak{G}_1$ . Let  $\Delta_1^* = \Delta_1 \cup [\mathfrak{G}_1 \setminus \mathcal{E}]$ . Fix a configuration  $\xi$  in each set  $\mathcal{E}_i, 1 \leq i \leq \kappa$ . We proved above and in the previous section that the triples  $(\mathcal{E}_i, \mathcal{E}_i \cup \Delta_1, \xi_i)$  are valleys. The next result states that we may increase  $\Delta_1$  to  $\Delta_1^*$ .

**Lemma 5.2.** The triples  $(\mathcal{E}_i, \mathcal{E}_i \cup \Delta_1^*, \xi_i), |\Lambda_L| < i \leq \kappa$ , are valleys of depth  $e^{\beta}/Z(\mathcal{E}_i)$ . Moreover, for every  $|\Lambda_L| < i \leq \kappa, 1 \leq j \neq i \leq \kappa, \xi \in \mathcal{E}_i$ ,

$$\lim_{\beta \rightarrow \infty} \mathbf{P}_{\xi}^{\beta} [H(\check{\mathcal{E}}_i) = H(\mathcal{E}_j)] = Q(\mathcal{E}_i, \mathcal{E}_j).$$

*Proof.* We have already remarked in (4.6) that it follows from claim (2) of the propositions of the previous section that the depth of the valleys  $(\mathcal{E}_i, \mathcal{E}_i \cup \Delta_1, \xi_i)$ ,  $|\Lambda_L| < i \leq \kappa$ , is  $e^\beta/Z(\mathcal{E}_i)$ . The first assertion of the lemma follows from Lemma 7.1 and from the fact proved in the previous section that for  $|\Lambda_L| < i \leq \kappa$ ,

$$\lim_{\beta \rightarrow \infty} \min_{\xi \in \mathcal{E}_i} \mathbf{P}_\xi^\beta [H(\Gamma \setminus \mathcal{E}_i) = H(\mathcal{E})] = 1.$$

The second statement of the lemma follows from the definition of the probability measure  $Q(\mathcal{E}_i, \cdot)$  introduced in the previous section.  $\square$

Denote by  $\{\eta_t^\mathcal{E} : t \geq 0\}$  the trace of the process  $\eta_t^\beta$  on  $\mathcal{E}$ . The jumps rates of the Markov process  $\eta_t^\mathcal{E}$  are represented by  $R_\beta^\mathcal{E}(\eta, \xi)$ . Fix an element  $\xi_i$  in each set  $\mathcal{E}_i$ .

**Proposition 5.3.** *The sequence of Markov processes  $\{\eta_t^\beta : t \geq 0\}$  exhibits a tunneling behavior on the time-scale  $e^\beta$ , with metastates  $\{\mathcal{E}_j : 1 \leq j \leq \kappa\}$ , metapoints  $\xi_j$ ,  $1 \leq j \leq \kappa$ , and asymptotic Markov dynamics characterized by the rates*

$$\begin{aligned} r(\mathcal{E}_i, \mathcal{E}_j) &= 0, \quad 1 \leq i \leq |\Lambda_L|, \quad 1 \leq j \neq i \leq \kappa, \\ r(\mathcal{E}_i, \mathcal{E}_j) &= Z(\mathcal{E}_i)Q(\mathcal{E}_i, \mathcal{E}_j), \quad |\Lambda_L| < i \leq \kappa, \quad 1 \leq j \neq i \leq \kappa. \end{aligned}$$

*Proof.* We check that the first two assumptions of [2, Theorem 2.7] are fulfilled. We start with assumption (H1). For the valleys  $\mathcal{E}_j$  which are singletons, there is nothing to prove. For the other ones, as  $\check{\mathcal{E}}_j \subset [\mathcal{E}_j \cup \Delta_1]^c$ , assumption (H1) follows from the proofs of Propositions 4.2 and 4.6.

We turn to assumption (H0). Denote by  $r_\beta(\mathcal{E}_i, \mathcal{E}_j)$  the average rates of the trace process:

$$r_\beta(\mathcal{E}_i, \mathcal{E}_j) = \frac{1}{\mu_\beta(\mathcal{E}_i)} \sum_{\eta \in \mathcal{E}_i} \mu_\beta(\eta) \sum_{\xi \in \mathcal{E}_j} R_\beta^\mathcal{E}(\eta, \xi).$$

We claim that  $e^\beta r_\beta(\mathcal{E}_i, \mathcal{E}_j)$ ,  $1 \leq i \neq j \leq \kappa$ , converges to a limit denoted by  $r(i, j)$ , and that  $\sum_{j \neq i} r(i, j) = 0$ ,  $1 \leq i \leq |\Lambda_L|$ ,  $\sum_{j \neq i} r(i, j) \in (0, \infty)$ ,  $i > |\Lambda_L|$ .

Consider first the case  $i > |\Lambda_L|$ . We may rewrite  $e^\beta r_\beta(\mathcal{E}_i, \mathcal{E}_j)$  as  $e^\beta r_\beta(\mathcal{E}_i, \check{\mathcal{E}}_i) \times [r_\beta(\mathcal{E}_i, \mathcal{E}_j)/r_\beta(\mathcal{E}_i, \check{\mathcal{E}}_i)]$ . By [4, Corollary 4.4],  $r_\beta(\mathcal{E}_i, \mathcal{E}_j)/r_\beta(\mathcal{E}_i, \check{\mathcal{E}}_i)$  converges to a number  $p(\mathcal{E}_i, \mathcal{E}_j) \in [0, 1]$ .

On the other hand, by [2, Lemma 6.7],  $e^\beta r_\beta(\mathcal{E}_i, \check{\mathcal{E}}_i) = e^\beta \text{cap}_\beta(\mathcal{E}_i, \check{\mathcal{E}}_i)/\mu_\beta(\mathcal{E}_i)$ . From the results stated in the previous section, it is easy to construct a path  $\gamma$  from  $\mathcal{E}_i$  to  $\check{\mathcal{E}}_i$  such that  $G_\beta(\gamma) = e^{-\beta} \mu_\beta(\eta)$ ,  $\eta \in \mathcal{E}_i$ . It is also easy to see that any path  $\gamma'$  from  $\mathcal{E}_i$  to  $\check{\mathcal{E}}_i$  is such that  $G_\beta(\gamma) \leq e^{-\beta} \mu_\beta(\eta)$ ,  $\eta \in \mathcal{E}_i$ . Hence,  $G_\beta(\mathcal{E}_i, \check{\mathcal{E}}_i) = e^{-\beta} \mu_\beta(\eta)$ ,  $\eta \in \mathcal{E}_i$ . Assumption (H0) for  $i > |\Lambda_L|$  follows from this identity and (3.1).

Fix now  $i \leq |\Lambda_L|$ . Since  $r_\beta(\mathcal{E}_i, \mathcal{E}_j) \leq r_\beta(\mathcal{E}_i, \check{\mathcal{E}}_i)$ , we have to show that the rescaled rate  $e^\beta r_\beta(\mathcal{E}_i, \check{\mathcal{E}}_i) = e^\beta \text{cap}_\beta(\mathcal{E}_i, \check{\mathcal{E}}_i)/\mu_\beta(\mathcal{E}_i)$  vanishes as  $\beta \uparrow \infty$ . Since  $G_\beta(\mathcal{E}_i, \check{\mathcal{E}}_i) \leq e^{-2\beta} \mu_\beta(\eta)$ ,  $\eta \in \mathcal{E}_i$ , the result follows from (3.1).

In view of the proof of [4, Lemma 10.2] and Lemma 5.2,  $e^\beta r_\beta(\mathcal{E}_i, \mathcal{E}_j)$ ,  $1 \leq i \neq j \leq \kappa$ , converges to  $Z(\mathcal{E}_i)Q(\mathcal{E}_i, \mathcal{E}_j)$ .

It remains to show property (M3) of tunneling, which states that the time spent outside  $\mathcal{E}$  is negligible. Fix  $1 \leq i \leq \kappa$  and  $\xi \in \mathcal{E}_i$ . Denote by  $\{H_j : j \geq 1\}$  the times of the successive returns to  $\mathcal{E}$ :  $H_1 = H^+(\mathcal{E})$ ,  $H_{j+1} = H^+(\mathcal{E}) \circ \theta_{H_j}$ ,  $j \geq 1$ . To prove

(M3), it is enough to show that

$$\lim_{k \rightarrow \infty} \lim_{\beta \rightarrow \infty} \mathbf{P}_\xi^\beta [H_k \leq te^\beta] = 0 \quad \text{and} \quad \lim_{\beta \rightarrow \infty} \mathbf{E}_\xi^\beta [e^{-\beta} \int_0^{H_k \wedge te^\beta} \mathbf{1}\{\eta_s^\beta \in \Delta_1^*\} ds] = 0 \quad (5.1)$$

for all  $k \geq 1$ .

Since  $H_1 = H^+(\mathcal{E})$  is greater than the time of the first jump, there exists a positive constant  $c_0$ , independent of  $\beta$ , which turns  $H_1 = H^+(\mathcal{E})$  bounded below by an exponential time of parameter  $c_0 e^\beta$ ,  $\mathbf{P}_\eta^\beta$  almost surely for all  $\eta \in \mathcal{E}$ . The first result of (5.1) follows from this observation and of the strong Markov property.

To estimate the second term of (5.1), fix  $k \geq 1$  and rewrite the time integral as  $\sum_{0 \leq j < k} \int_{H_j \wedge te^\beta}^{H_{j+1} \wedge te^\beta}$ . For a fixed  $j$ , the integral vanishes unless  $H_j < te^\beta$ . In this case, we may apply the strong Markov property to estimate the expectation by

$$k \sup_{\xi \in \mathcal{E}} \mathbf{E}_\xi^\beta [e^{-\beta} \int_0^{H_1 \wedge te^\beta} \mathbf{1}\{\eta_s^\beta \in \Delta_1^*\} ds] .$$

If  $\xi$  belongs to  $\mathcal{E}_i$ ,  $1 \leq i \leq |\Lambda_L|$ , the expectation is bounded above by  $t \mathbf{P}_\xi^\beta [\tau_1 \leq te^\beta]$ , where  $\tau_1$  is the time of the first jump. This expression vanishes because  $\tau_1$  is an exponential time whose mean is of order  $e^{2\beta}$ . For  $i > |\Lambda_L|$ , we have seen in the proofs of the propositions of the previous section that the time spent between two visits to  $\mathcal{E}$  can be estimated by the time a rate one, finite state, irreducible Markov process needs to visit a state. This concludes the proof of the proposition.  $\square$

Let  $\Psi : \mathcal{E} \rightarrow \{1, \dots, \kappa\}$  be the blind function  $\Psi(\eta) = \sum_{1 \leq j \leq \kappa} j \mathbf{1}\{\eta \in \mathcal{E}_j\}$ . It follows from the previous result that the non-Markovian process  $X_t^\beta = \Psi(\eta_{te^\beta}^\mathcal{E})$  converges to the Markov process on  $\{1, \dots, \kappa\}$  with jump rates  $r(i, j) = Z(\mathcal{E}_i)Q(\mathcal{E}_i, \mathcal{E}_j)$ . The states  $\{1, \dots, |\Lambda_L|\}$  are absorbing, while the states  $\{|\Lambda_L| + 1, \dots, \kappa\}$  are transient for the asymptotic dynamics.

Let  $q(i, j)$ ,  $1 \leq i \leq \kappa$ ,  $1 \leq j \leq |\Lambda_L|$ , be the probability that starting from  $i$  the asymptotic process eventually reaches the absorbing point  $j$ :

$$q(i, j) = \mathbb{P}_i[X_t = j \text{ for some } t > 0] , \quad (5.2)$$

where  $\mathbb{P}_i$  stands for the probability on the path space  $D([0, \infty), \{1, \dots, \kappa\})$  induced by the Markov process with rates  $r(j, k)$  starting from  $i$ . We sometimes denote  $q(i, j)$  by  $q(\mathcal{E}_i, \mathcal{E}_j)$ .

## 6. TUNNELING AMONG THE DEEP VALLEYS

We prove in this section the main result of this article. Recall that we denoted by  $\mathcal{E}_\mathbf{x}$ ,  $\mathbf{x} \in \Lambda_L$ , the singletons  $\{\eta^\mathbf{x}\}$ .

Denote by  $F_\mathbf{x}$  the set of configurations which can be reached from  $\eta^\mathbf{x}$  by a jump of rate  $e^{-2\beta}$ . The set  $F_\mathbf{x}$  has 8 elements.

**Lemma 6.1.** *Fix  $\mathbf{x} \in \Lambda_L$ . For each  $\xi \in F_\mathbf{x}$  there exists a probability measure  $A_0(\xi, \cdot)$  defined on  $\Gamma$  such that*

$$\lim_{\beta \rightarrow \infty} \mathbf{P}_\xi^\beta [\eta(H_\Gamma) \in \Pi] = A_0(\xi, \Pi) , \quad \Pi \subset \Gamma .$$

Moreover, for  $i = 1, 2$ ,

$$\begin{aligned} A_0(\sigma^{\mathbf{w}, \mathbf{w}-e_i} \eta^{\mathbf{w}}, \eta^{\mathbf{w}}) &= p(\mathbf{w} - e_i, \mathbf{w}, \partial Q_{\mathbf{w}}), \\ A_0(\sigma^{\mathbf{w}, \mathbf{w}-e_i} \eta^{\mathbf{w}}, \mathcal{E}_{\mathbf{w}}^{0,j}) &= p(\mathbf{w} - e_i, Q_{\mathbf{w}}^{0,j}, \partial Q_{\mathbf{w}}). \end{aligned}$$

*Proof.* To fix ideas, assume, without loss of generality, that  $\xi = \sigma^{\mathbf{w}, \mathbf{w}-e_2} \eta^{\mathbf{w}}$ . The particle at  $\mathbf{w} - e_2$  performs a rate one, symmetric random walk until it reaches the boundary of  $Q_{\mathbf{w}} \setminus \{\mathbf{w}\}$ . This time coincides with the time in which the process attains  $\Gamma$ . All other jumps have rate at most  $e^{-\beta}$  and can therefore be neglected.  $\square$

The values of  $A_0(\xi, \cdot)$ ,  $\xi \in F_{\mathbf{x}}$ , can be obtained from  $A_0(\sigma^{\mathbf{w}, \mathbf{w}-e_i} \eta^{\mathbf{w}}, \cdot)$  by symmetry. Recall from (5.2) the definition of the probability  $q(\mathcal{E}_j, \cdot)$ . Let

$$Z = \sum_{\xi \in F_{\mathbf{x}}} \sum_{j=1}^{\kappa} A_0(\xi, \mathcal{E}_j) q(\mathcal{E}_j, \check{\mathcal{F}}_{\mathbf{x}}) = \sum_{\xi \in F_{\mathbf{x}}} \sum_{j=1}^{\kappa} A_0(\xi, \mathcal{E}_j) [1 - q(\mathcal{E}_j, \mathcal{E}_{\mathbf{x}})],$$

where  $\check{\mathcal{F}}_{\mathbf{x}} = \cup_{\mathbf{y} \neq \mathbf{x}} \mathcal{E}_{\mathbf{y}}$ , the union being carried over  $\mathbf{y} \in \Lambda_L$ . Denote by  $\Delta$  the configurations which are not ground states:  $\Delta = \Omega_{L,K} \setminus \Omega^0$ , and let  $\mathcal{F} = \cup_{\mathbf{y}} \mathcal{E}_{\mathbf{y}}$ .

**Proposition 6.2.** *Fix  $\mathbf{x} \in \Lambda_L$ .*

- (1) *The triple  $(\mathcal{E}_{\mathbf{x}}, \mathcal{E}_{\mathbf{x}} \cup \Delta, \eta^{\mathbf{x}})$  is a valley of depth  $\mu_{\beta}(\eta^{\mathbf{x}})/\text{cap}_{\beta}(\mathcal{E}_{\mathbf{x}}, \check{\mathcal{F}}_{\mathbf{x}})$ ;*
- (2) *Under  $\mathbf{P}_{\eta^{\mathbf{x}}}^{\beta}$ ,  $H(\check{\mathcal{F}}_{\mathbf{x}})/e^{2\beta}$  converges in distribution to an exponential random variable of parameter  $Z$ ;*
- (3) *For any  $\mathbf{y} \neq \mathbf{x}$ ,*

$$\lim_{\beta \rightarrow \infty} \mathbf{P}_{\eta^{\mathbf{x}}}^{\beta} [\eta(H(\check{\mathcal{F}}_{\mathbf{x}})) = \eta^{\mathbf{y}}] = \frac{1}{Z} \sum_{\xi \in F_{\mathbf{x}}} \sum_{j=1}^{\kappa} A_0(\xi, \mathcal{E}_j) q(\mathcal{E}_j, \mathcal{E}_{\mathbf{y}}) =: \mathbb{Q}(\mathbf{x}, \mathbf{y}).$$

*Proof.* Recall [2, Theorem 2.4]. By definition of the set  $\Delta$ ,  $\mu_{\beta}(\Delta)/\mu_{\beta}(\mathcal{E}_{\mathbf{x}})$  is of order  $e^{-\beta}$ . Condition (2.15) is therefore fulfilled. Since  $\mathcal{E}_i$  is a singleton, condition (2.14) holds automatically and the result follows.

The proof of the second assertion is similar to the one of the second claim in Proposition 4.2 with the following modifications. We first need to replace the normalization  $e^{\beta}$  by  $e^{2\beta}$  and to define  $\tau_1$  as the time of the first jump, to write

$$H(\check{\mathcal{F}}_{\mathbf{x}}) = \tau_1 + H(\mathcal{F}) \circ \theta_{\tau_1} + \mathbf{1}\{H(\mathcal{F}) \circ \theta_{\tau_1} = H(\mathcal{E}_{\mathbf{x}}) \circ \theta_{\tau_1}\} H(\check{\mathcal{F}}_{\mathbf{x}}) \circ \theta_{H^+(\mathcal{F})}.$$

At this point, we repeat the arguments presented in the proof of Proposition 4.2. In the present context,  $\tau_1$  and  $\eta_{\tau_1}$  are independent by the Markov property, and  $\eta_{H(\mathcal{E}_{\mathbf{x}})} = \eta^{\mathbf{x}}$ . We may therefore skip the coupling arguments of Proposition 4.2.

In contrast, we need to show that

$$\lim_{A \rightarrow \infty} \lim_{\beta \rightarrow \infty} \max_{\zeta \in F_{\mathbf{x}}} \mathbf{P}_{\zeta}^{\beta} [H(\mathcal{F}) > Ae^{\beta}] = 0. \quad (6.1)$$

Starting from  $\zeta \in F_{\mathbf{x}}$ , in a time of order one the process reaches  $\mathcal{E}$ . It follows from Proposition 5.3 that once at  $\mathcal{E}$  in a time of order  $e^{\beta}$  the process reaches one of the absorbing point  $\{\eta^{\mathbf{x}} : \mathbf{x} \in \Lambda_L\}$  of the asymptotic Markovian dynamics characterized by the rates  $r(\cdot, \cdot)$ . This proves (6.1).

It follows from this result and the proof of Proposition 4.2 that to prove the second assertion of the proposition it is enough to show that

$$\lim_{\beta \rightarrow \infty} \sum_{\zeta \in F_{\mathbf{x}}} \mathbf{P}_{\zeta}^{\beta} [H(\mathcal{F}) \neq H(\mathcal{E}_{\mathbf{x}})] = Z.$$



Since  $H(\mathcal{E}) \leq \min\{H(\mathcal{F}), H(\mathcal{E}_\mathbf{x})\}$ , by the strong Markov property we may rewrite the previous probability as

$$\mathbf{E}_\zeta^\beta \left[ \mathbf{P}_{\eta(H(\mathcal{E}))}^\beta [H(\mathcal{F}) \neq H(\mathcal{E}_\mathbf{x})] \right].$$

We computed in Lemma 6.1 the asymptotic distribution of  $\eta(H(\mathcal{E}))$  and we represented by  $q(\mathcal{E}_j, \mathcal{E}_\mathbf{y})$  the probability that the asymptotic process starting from a set  $\mathcal{E}_j$ ,  $1 \leq j \leq \kappa$ , eventually reaches the absorbing state  $\mathcal{E}_\mathbf{y}$ ,  $\mathbf{y} \in \Lambda_L$ . The second assertion of the proposition follows from these two results.

We now turn to the third assertion of the proposition. Fix  $\mathbf{y} \neq \mathbf{x}$ . This argument is also similar to the one of Proposition 4.2. Denote by  $\{H_j : j \geq 1\}$  the successive return times to  $\mathcal{F}$ :

$$H_1 = H^+(\mathcal{F}), \quad H_{j+1} = H^+(\mathcal{F}) \circ \theta_{H_j}, \quad j \geq 1.$$

With this notation,

$$\mathbf{P}_{\eta^\mathbf{x}}^\beta [\eta(H(\check{\mathcal{F}}_\mathbf{x})) = \eta^\mathbf{y}] = \sum_{j \geq 1} \mathbf{P}_{\eta^\mathbf{x}}^\beta [\eta(H_k) = \eta^\mathbf{x}, 1 \leq k \leq j-1, \eta(H_j) = \eta^\mathbf{y}]. \quad (6.2)$$

By the strong Markov property, if  $\tau_1$  stands for the time of the first jump, for any  $\mathbf{z} \in \Lambda_L$ ,

$$\mathbf{P}_{\eta^\mathbf{x}}^\beta [\eta(H_1) = \eta^\mathbf{z}] = \mathbf{E}_{\eta^\mathbf{x}}^\beta \left[ \mathbf{E}_{\eta_{\tau_1}}^\beta \left[ \mathbf{P}_{\eta_{H(\mathcal{E})}}^\beta [\eta(H_\mathcal{F}) = \eta^\mathbf{z}] \right] \right].$$

As  $\beta \uparrow \infty$ , this expression converges to

$$\frac{1}{8} \sum_{\xi \in F_\mathbf{x}} \sum_{j=1}^{\kappa} A_0(\xi, \mathcal{E}_j) q(\mathcal{E}_j, \mathcal{E}_\mathbf{z}).$$

The third assertion of the proposition follows from (6.2), this identity and the strong Markov property.  $\square$

It follows from (1) and (2) that the triple

$$(\mathcal{E}_\mathbf{x}, \mathcal{E}_\mathbf{x} \cup \Delta, \eta^\mathbf{x}) \text{ is in fact a valley of depth } e^{2\beta}/Z. \quad (6.3)$$

**Corollary 6.3.** *The sequence of Markov processes  $\{\eta_t^\beta : t \geq 0\}$  exhibits a tunneling behavior on the time-scale  $e^{2\beta}$ , with metastates  $\{\mathcal{E}_\mathbf{x} : \mathbf{x} \in \Lambda_L\}$ , metapoints  $\{\eta^\mathbf{x}\}$  and asymptotic Markov dynamics characterized by the rates*

$$r(\mathcal{E}_\mathbf{x}, \mathcal{E}_\mathbf{y}) = Z \mathbb{Q}(\mathbf{x}, \mathbf{y}), \quad \mathbf{x} \neq \mathbf{y} \in \Lambda_L.$$

*Proof.* The proof is similar to the one of Proposition 5.3. We first check that assumptions (H0) and (H1) of [2, Theorem 2.7] are fulfilled. Hypothesis (H1) is trivially satisfied since the sets  $\mathcal{E}_\mathbf{x}$  are singletons.

To prove assumption (H0), denote  $\{\eta_t^\mathcal{F} : t \geq 0\}$  the trace of the process  $\eta_t^\beta$  on  $\mathcal{F}$ , and by  $R_\beta^\mathcal{F}$  the jump rates of the trace process. Note that in this case of singleton valleys, the average rates coincide with the rates. We claim that  $e^{2\beta} R_\beta^\mathcal{F}(\mathcal{E}_\mathbf{x}, \mathcal{E}_\mathbf{y})$ ,  $\mathbf{x} \neq \mathbf{y} \in \Lambda_L$ , converges to a limit denoted by  $R(\mathbf{x}, \mathbf{y})$ .

We may rewrite  $e^{2\beta} R_\beta^\mathcal{F}(\mathcal{E}_\mathbf{x}, \mathcal{E}_\mathbf{y})$  as  $e^{2\beta} R_\beta^\mathcal{F}(\mathcal{E}_\mathbf{x}, \check{\mathcal{F}}_\mathbf{x}) \times [R_\beta^\mathcal{F}(\mathcal{E}_\mathbf{x}, \mathcal{E}_\mathbf{y}) / R_\beta^\mathcal{F}(\mathcal{E}_\mathbf{x}, \check{\mathcal{F}}_\mathbf{x})]$ . By [4, Corollary 4.4],  $R_\beta^\mathcal{F}(\mathcal{E}_\mathbf{x}, \mathcal{E}_\mathbf{y}) / R_\beta^\mathcal{F}(\mathcal{E}_\mathbf{x}, \check{\mathcal{F}}_\mathbf{x})$  converges to a number  $p(\mathcal{E}_\mathbf{x}, \mathcal{E}_\mathbf{y}) \in [0, 1]$ . On the other hand, by [2, Lemma 6.7],  $e^{2\beta} R_\beta^\mathcal{F}(\mathcal{E}_\mathbf{x}, \check{\mathcal{F}}_\mathbf{x}) = e^{2\beta} \text{cap}_\beta(\mathcal{E}_\mathbf{x}, \check{\mathcal{F}}_\mathbf{x}) / \mu_\beta(\mathcal{E}_\mathbf{x})$ . Clearly,  $G_\beta(\mathcal{E}_\mathbf{x}, \check{\mathcal{F}}_\mathbf{x}) = e^{-2\beta} \mu_\beta(\eta^\mathbf{x})$ . Hence, assumption (H0) follows from (3.1).

In view of [4, Lemma 10.2], Proposition 6.2 and (6.3),  $e^{2\beta} R_\beta^\mathcal{F}(\mathcal{E}_\mathbf{x}, \mathcal{E}_\mathbf{y})$ ,  $\mathbf{x} \neq \mathbf{y} \in \Lambda_L$ , converges to  $Z\mathbb{Q}(\mathbf{x}, \mathbf{y})$ .

It remains to show property (M3) of tunneling, which states that the time spent outside  $\mathcal{F}$  is negligible. Fix  $\mathbf{x} \in \Lambda_L$ . Denote by  $\{H_j : j \geq 1\}$  the times of the successive returns to  $\mathcal{F}$ :  $H_1 = H^+(\mathcal{F})$ ,  $H_{j+1} = H^+(\mathcal{F}) \circ \theta_{H_j}$ ,  $j \geq 1$ . To prove (M3), it is enough to show that

$$\begin{aligned} \lim_{k \rightarrow \infty} \lim_{\beta \rightarrow \infty} \mathbf{P}_{\eta^\mathbf{x}}^\beta [H_k \leq te^{2\beta}] &= 0 \quad \text{and} \\ \lim_{\beta \rightarrow \infty} \mathbf{E}_{\eta^\mathbf{x}}^\beta \left[ e^{-2\beta} \int_0^{H_k \wedge te^{2\beta}} \mathbf{1}\{\eta_s^\beta \in \Delta\} ds \right] &= 0 \end{aligned} \quad (6.4)$$

for all  $k \geq 1$ .

Since  $H_1 = H^+(\mathcal{F})$  is greater than the time of the first jump,  $H_1$  is bounded below by an exponential time of parameter  $8e^{-2\beta}$ ,  $\mathbf{P}_\eta^\beta$  almost surely for all  $\eta \in \mathcal{E}$ . The first line of (6.4) follows from this observation and from the strong Markov property.

To estimate the second term of (6.4), fix  $k \geq 1$  and rewrite the time integral as  $\sum_{0 \leq j < k} \int_{H_j \wedge te^{2\beta}}^{H_{j+1} \wedge te^{2\beta}}$ . For a fixed  $j$ , the integral vanishes unless  $H_j < te^{2\beta}$ . Hence, by the strong Markov property, the expectation is less than or equal to

$$k \max_{\mathbf{y} \in \Lambda_L} \mathbf{E}_{\eta^\mathbf{y}}^\beta \left[ e^{-2\beta} \int_0^{H_1 \wedge te^{2\beta}} \mathbf{1}\{\eta_s^\beta \in \Delta\} ds \right].$$

Recall that we denoted by  $F_\mathbf{y}$  the set of configurations which can be reached from  $\eta^\mathbf{y}$  by a jump of rate  $e^{-2\beta}$ . By the strong Markov property, this expression is bounded by

$$k \max_{\mathbf{y} \in \Lambda_L} \max_{\xi \in F_\mathbf{y}} \mathbf{E}_\xi^\beta [e^{-2\beta} H(\mathcal{F}) \wedge t].$$

By (6.1) this expression vanishes as  $\beta \uparrow \infty$ .  $\square$

## 7. GENERAL RESULTS

We prove in this section an useful general result. Fix a sequence  $(E_N : N \geq 1)$  of countable state spaces. The elements of  $E_N$  are denoted by the Greek letters  $\eta, \xi$ . For each  $N \geq 1$  consider a matrix  $R_N : E_N \times E_N \rightarrow \mathbb{R}$  such that  $R_N(\eta, \xi) \geq 0$  for  $\eta \neq \xi$ ,  $-\infty < R_N(\eta, \eta) \leq 0$  and  $\sum_{\xi \in E_N} R_N(\eta, \xi) = 0$  for all  $\eta \in E_N$ .

Let  $\{\eta_t^N : t \geq 0\}$  be the *minimal* right-continuous Markov process associated to the jump rates  $R_N(\eta, \xi)$  [23]. It is well known that  $\{\eta_t^N : t \geq 0\}$  is a strong Markov process with respect to the filtration  $\{\mathcal{F}_t^N : t \geq 0\}$  given by  $\mathcal{F}_t^N = \sigma(\eta_s^N : s \leq t)$ . Let  $\mathbf{P}_\eta$ ,  $\eta \in E_N$ , be the probability measure on  $D(\mathbb{R}_+, E_N)$  induced by the Markov process  $\{\eta_t^N : t \geq 0\}$  starting from  $\eta$ .

Consider two sequences  $\mathcal{W} = (W_N \subseteq E_N : N \geq 1)$ ,  $\mathcal{B} = (B_N \subseteq E_N : N \geq 1)$  of subsets of  $E_N$ , the second one containing the first and being properly contained in  $E_N$ :  $W_N \subseteq B_N \subsetneq E_N$ . Fix a point  $\boldsymbol{\xi} = (\xi_N \in W_N : N \geq 1)$  in  $\mathcal{W}$  and a sequence of positive numbers  $\boldsymbol{\theta} = (\theta_N : N \geq 1)$ .

Next result states an obvious fact. We may add to the basin  $\mathcal{B}$  of a valley  $(\mathcal{W}, \mathcal{B}, \boldsymbol{\xi})$  a set  $\mathcal{C}$  never visited by the process without modifying the properties of the valley.

**Lemma 7.1.** *Assume that the triple  $(\mathcal{W}, \mathcal{B}, \xi)$  is a valley of depth  $\theta$  and attractor  $\xi$ . Let  $\mathcal{C} = (C_N \subset E_N : N \geq 1)$  be a sequence of sets such that  $B_N^c$  is attained before  $C_N$  when starting from  $W_N$ :*

$$\lim_{N \rightarrow \infty} \inf_{\eta \in W_N} \mathbf{P}_\eta [H_{B_N^c} < H_{C_N}] = 1. \quad (7.1)$$

*Then, the triple  $(\mathcal{W}, \mathcal{B} \cup \mathcal{C}, \xi)$  is a valley of depth  $\theta$  and attractor  $\xi$ .*

*Proof.* We have to check the three conditions of [2, Definition 2.1]. The first one is obvious because  $B_N^c \supset (B_N \cup C_N)^c$ . On the event  $\{H_{B_N^c} < H_{C_N}\}$ ,  $H_{B_N^c} = H_{(B_N \cup C_N)^c}$ . Hence, the convergence in distribution of  $H_{(B_N \cup C_N)^c}/\theta_N$  to a mean one exponential variable follows from (7.1) and from the one of  $H_{B_N^c}/\theta_N$ . For the same reasons, on the set  $\{H_{B_N^c} < H_{C_N}\}$ ,  $\int_0^{H_{B_N^c}} \mathbf{1}\{\eta_s^N \in A\} ds = \int_0^{H_{(B_N \cup C_N)^c}} \mathbf{1}\{\eta_s^N \in A\} ds$ . In particular, property (V3) for the triple  $(\mathcal{W}, \mathcal{B} \cup \mathcal{C}, \xi)$  follows from (7.1) and (V3) for the valley  $(\mathcal{W}, \mathcal{B}, \xi)$ .  $\square$

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